V. Z. Parton, P. I. Perlin

Mathematical Methods of the Theory of Elasticity

Elasticity

Mir Publishers

This monograph contains a modern course in the mathematical theory of elasticity. Unlike the prevalent approach towards constructing such a course, the structure of this book is such that each chapter is devoted to some mathematical method and its applications to specific problems in the theory of elasticity. Such an approach helps in concentrating the reader's attention on the basic questions from the mathematical point of view. This is in accordance with the present-day state of affairs in the theory of elasticity, which is considered to be a specialized branch of computational mathematics.

For the convenience of beginners, Vol. 1 (Chapters 1—3) contains mathematical topics necessary for understanding the main aspects of this course, as well as the basic concepts in the theory of elasticity. Besides, certain types of problems of the theory of elasticity have been considered. Their solution is reduced to the traditional classical problems of the theory of partial differential equations.

The second volume (Chapters 4—8) is devoted to a description of the following methods: the method of separation of variables, the method of analytic functions, the method of integral transformations, the method of potentials, the variational methods, and the difference methods. Considerable attention has been paid to the questions concerning the numerical realization of these methods and their applications to the problems in the theory of elasticity. In order to illustrate the effectiveness of these methods, a wide range of scientific publications of recent years has been employed.

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Mathematical Methods of the Theory of Elasticity

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Prof. Vladimir Parton, D.Sc., Corresponding Member of the International Academy of Astronautics, is one of the leading Soviet scientists in the field of mechanics of a deformable rigid body and the strength of materials in machine building.
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Pyotr Perlin, D.Sc., is on the staff of the Moscow Physico-Technical Institute. Besides teaching the theory of elasticity and plasticity, Dr. Perlin delivers special courses on the branches of mathematics which are applied in the theory of elasticity.
Pyotr Perlin is the author of over 60 scientific papers on various aspects of the mechanics of a rigid deformable solid, and especially on the construction of computer algorithms. He has proposed a new computational method of solving elastic-plastic problems for an ideal plastic medium. Besides, he has produced computer programmes for an ideal solidifying medium.
During the last few years, Dr. Perlin has developed a new effective computational method for solving singular integral equations of basic three-dimensional problems in the theory of elasticity.

ИЗДАТЕЛЬСТВО · МИР ·

В. З. ПАРТОН, П. И. ПЕРЛИН

МЕТОДЫ МАТЕМАТИЧЕСКОЙ ТЕОРИИ УПРУГОСТИ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

V. Z. PARTON, P. I. PERLIN

Mathematical Methods of the Theory of Elasticity

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Contents

Preface

This is the second volume of the monograph "Mathematical Methods of the Theory of Elasticity". The first volume, consisting of three chapters, included basic mathematical concepts used in the theory of elasticity, the main ideas of the theory of elasticity, and formulation of basic boundary value problems. Besides, some general problems of the theory of elasticity were discussed, including the reduction of the boundary value problems of the theory of elasticity to classical boundary value problems of mathematical physics, an investigation of the peculiarities of the solution in the vicinity of singular points of the boundary, Saint-Venant's principle and its applications in the formulation of two-dimensional problems of the theory of elasticity, etc.

The second volume contains the main part of the monograph. The structure of this book, in accordance with its title, basically differs from the prevalent literature on the theory of elasticity which is classified according to the problems (bending and torsion of rods, the plane problem, the three-dimensional problem, etc.) and not according to the mathematical methods employed for solving them. The reverse approach, which was one of the compelling motives behind the writing of this book, helps in drawing the reader's attention to the very methods of solving problems. This is in keeping with the modern views which regard the theory of elasticity as a special applied branch of mathematical physics.

It should be noted that while systematizing this course on the theory of elasticity in accordance with the mathematical methods, the authors did not strive to attain any kind of uniformity while describing the material of different chapters. In case a full-fledged theory exists, it has been described together with some illustrative examples (the chapters describing the theory of analytical functions and the potential theory belong to this category). On the contrary, the main stress in other cases has been laid on the solution of specific problems. The reason for this approach (adopted, for example, in the chapter on separation of variables) is that this comparatively easy method is fully clarified in the beginning of this treatise (in Ch. 1), and hence only such specific problems of the theory of elasticity are of any interest where important and constructive results can be achieved. A similar situation arises in Ch. 6 (Integral Representation and Transformations), though for quite different reasons. Since no universal methods are available for solving this type of problems, the mathematical apparatus can be developed only for specific problems. While

selecting these problems, the authors were guided not only by the above-mentioned general criteria, but also by the novelty and originality of mathematical results, the importance of the problem for some field related to the theory of elasticity (for example, for fracture mechanics), and the possibility of obtaining general qualitative conclusions from the solutions of these problems.

The chapter dealing with variational and difference methods (Ch. 8) has also been written in an illustrative manner, and is based on the solution of specific problems. This is so because the variational methods and, in particular, the difference methods, form one of the most exhaustively investigated areas of computational mathematics (even from the point of view of their application to the problems of the theory of elasticity). Hence a detailed description of these methods is not possible here in view of the limitations of space. However, the examples which have been chosen in this chapter for solving specific problems of the theory of elasticity amply demonstrate the advantages and drawbacks of these methods.

The appendices contain a brief account of certain problems of continuum mechanics (whose constructive analysis is possible on the basis of linear theory of elasticity), as well as some problems of the linear theory of elasticity for an anisotropic medium.

V. Z. Parton, P. I. Perlin

Moscow, January, 1980

Chapter Four

The Methods of Separation of Variables in the Problems of the Theory of Elasticity

Section 1

Problems in the Theory of Elasticity for a Sphere and a Space with a Spherical Cavity

It was shown in Sec. 10, Ch. 1, Vol. 1 that a solution of the Dirichlet problem for a sphere may be obtained by the method of separation of variables involving adjoint sphericial functions. Recalling that in Sec. 5, Ch. 3, Vol. 1 it was established that the solution of the first fundamental problem in the theory of elasticity for a sphere may be reduced to the solution of three Dirichlet problems, we can directly realize the method of separation of variables for solving problems of the theory of elasticity as well (for the second fundamental problem, the same reasoning applies, though the calculations are more cumbersome). We shall apply the method of separation of variables using the Papkovich-Neuber representations, for solving the problem for a sphere. To begin with, we shall find the solution of an axially symmetric problem, which can then be used to construct Green's function for any loading problem.

It was established in Sec. 5, Ch. 3, Vol. 1 that in the case of axial symmetry, the displacements in spherical coordinates may be represented in terms of the projections of a certain harmonic vector $\psi(\psi_x, \psi_y, \psi_z)$ and a scalar harmonic funcion φ:

$$\begin{aligned} u_r &= 4(1-\nu)\psi_r - \frac{\partial}{\partial r}(r\psi_r + \varphi), \\ u_\theta &= 4(1-\nu)\psi_\theta - \frac{1}{r}\frac{\partial}{\partial \theta}(r\psi_r + \varphi), \end{aligned} \tag{1.1}$$

where ψ_r and ψ_θ are the projections of the vector ψ onto the axes r and θ.

We shall use these representations to obtain convenient representations (from the point of view of solving boundary value problems) of particular solutions of problems in the theory of elasticity for a sphere and a space with a spherical cavity. In order to construct the harmonic functions mentioned above, we shall use the method of separation of variables. Let us take a positive integer n. It follows from the discussion in Sec. 10, Ch. 1, Vol. 1 that in view of axial symmetry, the projections of the vector ψ onto the coordinate axes x and y may be chosen in the following form (A = const):

$$\begin{aligned} \psi_x &= Ar^n \cos\varphi \frac{dP_n(\cos\theta)}{d\theta}, \\ \psi_y &= Ar^n \sin\varphi \frac{dP_n(\cos\theta)}{d\theta}. \end{aligned} \tag{1.2}$$

The projection ψ_z is given by ($A^* = \text{const}$)

$$\psi_z = A^* r^n P_n(\cos\theta). \tag{1.3}$$

This leads to the following representation for the projections ψ_r and ψ_θ:

$$\begin{aligned}\psi_r &= r^n \left[A \sin\theta \frac{dP_n(\cos\theta)}{d\theta} + A^* \cos\theta P_n(\cos\theta) \right],\\ \psi_\theta &= r^n \left[A \cos\theta \frac{dP_n(\cos\theta)}{d\theta} - A^* \sin\theta P_n(\cos\theta) \right].\end{aligned} \tag{1.4}$$

These representations can be transformed with the help of the well-known identities [1]

$$(\mu^2 - 1)P_n'(\mu) = n(\mu P_n - P_{n-1}), \qquad \mu P_n'(\mu) = nP_n + P_{n-1}'. \tag{1.5}$$

In order to obtain simpler expressions, we put $A^* = -nA$. This gives

$$\begin{aligned}\psi_r &= -nAr^n P_{n-1}(\mu),\\ \psi_\theta &= -Ar^n P_{n-1}'(\mu)\sin\theta = Ar^n \frac{dP_{n-1}}{d\theta}.\end{aligned} \tag{1.6}$$

The function φ is taken in the form

$$\varphi = -Br^{n-1}P_{n-1}(\mu). \tag{1.7}$$

With the help of Eqs. (1.6) and (1.7), we can obtain the expressions for displacements and stresses inside the sphere in a fairly compact form. Thus, for any values of the constants A and B, we get the corresponding particular solutions which can be used for solving the axially symmetric boundary value problem for the case of a sphere.

A similar reasoning can be applied if n is negative. The same result can be obtained in this case by replacing n with $-(n + 1)$, and taking into account the identity $P_{-(n+1)} = P_n$. This identity follows from the fact that the equation for the Legendre polynomials is determined by the number $n(n + 1)$ and, consequently, is invariant with respect to the substitution $n' = -(n + 1)$. The particular solutions thus obtained may be used for solving axially symmetric problems in the case of a space with a spherical cavity.

Of course, the method of separation of variables could have been directly applied to Lamé's equations as well (in spherical coordinates and taking into account the axial symmetry). However, the method employing the Papkovich-Neuber representation is more convenient.

We shall now give the complete system of particular solutions (in terms of displacements and stresses) for the internal and external problems, obtained on the basis of the representations (1.6) and (1.7) [2]. For the internal problem we have

$$\begin{aligned}u_r &= [Ar^{n+1}(n + 1)(n - 2 + 4\nu) + Br^{n-1}n]P_n(\cos\theta),\\ u_\theta &= \left[Ar^{n+1}(n + 5 - 4\nu) + Br^{n-1}\right] \frac{dP_n(\cos\theta)}{d\theta};\end{aligned} \tag{1.8}$$

$$\frac{1}{2\mu}\sigma_r = \left[A(n+1)(n^2-n-2-2\nu)r^n + Bn(n-1)r^{n-2}\right]P_n(\cos\theta),$$

$$\frac{1}{2\mu}\tau_{r\theta} = \left[A(n^2+2n-1+2\nu)r^n + B(n-1)r^{n-2}\right]\frac{dP_n(\cos\theta)}{d\theta},$$

$$\frac{1}{2\mu}\sigma_\theta = -\left[A(n^2+4n+2+2\nu)(n+1)r^n + Bn^2r^{n-2}\right]P_n(\cos\theta) - \left[A(n+5-4\nu)r^n + Br^{n-2}\right]\frac{dP_n(\cos\theta)}{d\theta}\cot\theta, \tag{1.9}$$

$$\frac{1}{2\mu}\sigma_\varphi = \left[A(n+1)(n+2-2\nu-4n\nu)r^n + Bnr^{n-2}\right]P_n(\cos\theta) + \left[A(n+5-4\nu)r^n + Br^{n-2}\right]\frac{dP_n(\cos\theta)}{d\theta}\cot\theta.$$

For the external problem we have similar expressions for displacements and stresses:

$$u_r = [Cr^{-n}n(n+3-4\nu) - D(n+1)r^{-n-2}]P_n(\cos\theta),$$

$$u_\theta = \left[Cr^{-n}(-n+4-4\nu) + Dr^{-n-2}\right]\frac{dP_n(\cos\theta)}{d\theta}; \tag{1.10}$$

$$\frac{1}{2\mu}\sigma_r = \left[-Cnr^{-n-1}(n^2+3n-2\nu) + D(n+1)(n+2)r^{-n-3}\right]P_n(\cos\theta),$$

$$\frac{1}{2\mu}\tau_{r\theta} = \left[Cr^{-n-1}(n^2-2+2\nu) - D(n+2)r^{-n-3}\right]\frac{dP_n(\cos\theta)}{d\theta},$$

$$\frac{1}{2\mu}\sigma_\theta = \left[Cnr^{-n-1}(n^2-2n-1+2\nu) - D(n+1)^2r^{-n-3}\right]P_n(\cos\theta) - \left[Cr^{-n-1}(-n+4-4\nu) + Dr^{-n-3}\right]\frac{dP_n(\cos\theta)}{d\theta}\cot\theta, \tag{1.11}$$

$$\frac{1}{2\mu}\sigma_\varphi = \left[Cnr^{-n-1}(n+4-4n\nu-2\nu) - D(n+1)r^{-n-3}\right]P_n(\cos\theta) + \left[Cr^{-n-1}(-n+4-4\nu) + Dr^{-n-3}\right]\frac{dP_n(\cos\theta)}{d\theta}\cot\theta.$$

We shall denote by $K_n^\pm$ and $L_n^\pm$, $M_n^\pm$ and $N_n^\pm$ the expressions in square brackets in the representations for the displacements u_r and u_θ and for the stresses σ_r and $\tau_{r\theta}$ on the boundary, i.e. for $r \to R$ (the plus and minus signs correspond to the internal and the external problem respectively).

Let us use the representations obtained above for solving the first and the second internal and external problems. For the first problem we shall assume that the displacements are given:

$$u_r(\theta) = u_r^0(\theta), \quad u_\theta(\theta) = u_\theta^0(\theta) \quad (r = R), \tag{1.12}$$

while for the second problem we assume that the stresses are given:

$$\sigma_r(\theta) = \sigma_r^0(\theta), \quad \tau_{r\theta}(\theta) = \tau_{r\theta}^0(\theta) \quad (r = R). \tag{1.13}$$

Then we can calculate the unknown coefficients from the boundary conditions (1.12) and (1.13), using the series expansions in the Legendre polynomials:

$$K_n^\pm = \frac{2n+1}{2}\int_0^\pi u_r^0(\theta) P_n(\cos\theta)\sin\theta d\theta,$$

$$L_n^\pm = \frac{2n+1}{2}\int_0^\pi u_\theta^0(\theta)\frac{dP_n(\cos\theta)}{d\theta}\sin\theta d\theta; \tag{1.14}$$

$$M_n^\pm = \frac{2n+1}{2}\int_0^\pi \sigma_r^0(\theta) P_n(\cos\theta)\sin\theta d\theta,$$

$$N_n^\pm = \frac{2n+1}{2n(n+1)}\int_0^\pi \tau_{r\theta}^0(\theta)\frac{dP_n(\cos\theta)}{d\theta}\sin\theta d\theta. \tag{1.15}$$

After that we can reconstruct the required coefficients A_n, B_n, C_n, and D_n.

Let us now consider the solvability of algebraic systems of equations thus obtained. We shall start with the second internal problem. It follows from the first two formulas of (1.9) that the constants A_n and B_n can be determined when the corresponding determinant of the system differs from zero. In the case under consideration the determinant of the system

$$-2(n-1)[n(n-1) + (1+\nu)(2n+1)]R^{2(n-1)} \tag{1.16}$$

differs from zero for $n = 0, 2, 3, \ldots$, and therefore for these values all the equations are solvable. For $n = 1$ the solvability of the corresponding equations must follow from the equilibrium condition for a sphere, which is necessary for formulating the boundary value problem. We shall determine the principal vector P_z of the forces applied to a body (it follows directly from the symmetry conditions that the vector moment is equal to zero):

$$P_z = 2\pi R^2 \int_0^\pi [\sigma_r^0(\theta)\cos\theta - \tau_{r\theta}^0(\theta)\sin\theta]\sin\theta d\theta$$

$$= 2\pi R^2 \sum_{n=0}^\infty \int_{-1}^1 \{M_n \cos\theta P_n(\cos\theta)$$

$$- nN_n^+[\cos\theta P_n(\cos\theta) - P_{n-1}(\cos\theta)]\} d(\cos\theta) = 0. \tag{1.17}$$

Only the second term in this expressión is non-zero. As a result, we get

$$P_z = 2\pi R^2 \left[M_1^+ \int_{-1}^{1} \cos^2\theta d(\cos\theta) - N_1^+ \int_{-1}^{1} (\cos^2\theta - 1) d(\cos\theta) \right]$$

$$= \frac{4\pi R^2}{3}(M_1^+ + 2N_1^+). \tag{1.18}$$

Consequently, the equilibrium condition is equivalent to the equation

$$M_1^+ + 2N_1^+ = 0, \tag{1.19}$$

which enables us to assume that the equation is also solvable when the determinant is equal to zero. Taking this into account, we get

$$A_1 = -\frac{M_1^+}{8\mu R(1+\nu)} = \frac{N_1^+}{4\mu R(1+\nu)}. \tag{1.20}$$

The solution does not contain the coefficient B_1, which was formally introduced earlier in order to unify the notation in the expressions (1.8) and (1.9). It may turn out that the condition (1.19) may not be strictly satisfied on account of computational errors in the values of M_1^+ and N_1^+. This, however, should not deter us from using the formula (1.20) in its first or second version.

Similarly, by using the first two formulas (1.11) in the case of the external boundary value problem, we get a system of equations whose determinant is given by

$$2(n+2)[(n+2)(n+1) - (1+\nu)(2n+1)]R^{-2n-1}. \tag{1.21}$$

For $n = 0, 1, 2, \ldots$ this expression has a non-zero value.

The above approach can be easily extended to the case of equilibrium of a hollow concentric sphere.

As it often happens in the case when the method of separation of variables is applied, the solution in all the cases discussed above can be represented in the form of a series whose coefficients are integrals of the boundary conditions. If the order of summation and integration is interchanged in these series, it is possible (as in the harmonic case) to find the sum of the internal series, and this gives a compact representation of the solution. Following [3], we shall adopt this procedure for the case of the second internal problem (here, it is assumed that the shearing stresses vanish). In this case it is useful to employ the equality

$$P_n(\cos\theta)P_n(\cos\alpha) = \frac{2}{\pi} \int_0^{\pi/2} P_n(\lambda)\, d\psi, \tag{1.22}$$

where $\lambda = \cos(\theta + \alpha) + 2\sin\theta \sin\alpha \sin^2\psi$. This leads to an expression for the displacements u_r and u_θ in the form of double integrals:

$$u_r = \frac{R}{4\pi\mu} \int_0^{\pi} \sigma_r^0(\alpha)\sin\alpha\, d\alpha \int_0^{\pi/2} d\psi$$

$$\times \left\{ \sum_{n=2}^{\infty} P_n(\lambda) \left[A_{1n} \left(\frac{r}{R}\right)^{n+1} + A_{2n} \left(\frac{r}{R}\right)^{n-1} \right] + \frac{2(1-2\nu)}{1+\nu} \frac{r}{R} \right\},$$

$$(1.23)$$

$$u_\theta = \frac{R}{4\pi\mu} \int_0^\pi \sigma_r^0(\alpha) \sin\alpha \, d\alpha \frac{\partial}{\partial\theta} \int_0^{\pi/2} d\psi$$

$$\times \sum_{n=2}^{\infty} P_n(\lambda) \left[A_{3n} \left(\frac{r}{R}\right)^{n+1} + A_{4n} \left(\frac{r}{R}\right)^{n-1} \right];$$

$$A_{1n} = -\frac{(2n+1)(n-2+4\nu)(n+1)}{n^2+(1+2\nu)n+1+\nu},$$

$$A_{2n} = \frac{(2n+1)(n^2+2n-1+2\nu)n}{(n-1)[n^2+(1+2\nu)n+1+\nu]},$$

$$(1.24)$$

$$A_{3n} = -\frac{(2n+1)(n+5-4\nu)}{n^2+(1+2\nu)n+1+\nu},$$

$$A_{4n} = \frac{(2n+1)(n^2+2n-1+2\nu)}{(n-1)[n^2+(1+2\nu)n+1+\nu]}.$$

Thus, it is possible to transform the series appearing in the integrands in such a way as to obtain the expressions for the displacements in the closed form. For this purpose, we can express the rational fractions appearing in (1.24) as sums of polynomials and simplest rational fractions

$$A_{1n} = -(2n+1) + 4(1-\nu) + \frac{P}{n-n_1} + \frac{\bar{P}}{n-\bar{n}_1},$$

$$A_{2n} = (2n+1) + 4(1-\nu) + \frac{2}{n-1} + \frac{Q}{n-n_1} + \frac{\bar{Q}}{n-\bar{n}_1},$$

$$(1.25)$$

$$A_{3n} = -2 + \frac{S}{n-n_1} + \frac{\bar{S}}{n-\bar{n}_1},$$

$$A_{4n} = 2 + \frac{2}{n-1} + \frac{T}{n-n_1} + \frac{\bar{T}}{n-\bar{n}_1}.$$

Here, n_1 and $\bar{n}_1$ are complex conjugate roots of the equation

$$n^2 + (1+2\nu)n + 1 + \nu = 0. \qquad (1.26)$$

The constants P, Q, S, and T appearing in Eqs. (1.25) depend only on Poisson's ratio and can be easily determined.

Thus, the problem is reduced to the calculation of the sums of the form

$$\sum r^n P_n(\lambda), \quad \sum n r^n P_n(\lambda), \quad \sum \frac{r^n}{n-a} P_n(\lambda).$$

The value of the first sum is known [4]:

$$\sum_{n=0}^{\infty} r^n P_n(\lambda) = \frac{1}{s}, \quad s = \sqrt{r^2 - 2r\lambda + 1}.$$

The following relations are also valid:

$$\sum_{n=0}^{\infty} n r^n P_n(\lambda) = r \frac{\partial}{\partial r} \sum_{n=0}^{\infty} r^n P_n(\lambda) = \frac{r(r-\lambda)}{(r^2 - 2r\lambda + 1)^{3/2}},$$

$$\sum_{n=0}^{\infty} \frac{r^n}{n-a} P_n(\lambda) = r^a \int_0^r dr \sum_{n=0}^{\infty} r^{n-a-1} P_n(\lambda)$$

$$= \int_0^1 \frac{dy}{y^{1+a}\sqrt{r^2y^2 - 2ry\lambda + 1}}.$$

For $a = n_1$ and $\bar{n}_1$, the second integral coincides with a hypergeometric function to within the factor $1/a$. The integrals over the variable ψ can now be easily evaluated. The final expressions for displacements are given by

$$u_r(r,\theta) = \frac{R}{2\pi\mu} \int_0^\pi \sigma_r^0(\alpha) H_r\left(\frac{r}{R}, \theta, \alpha\right) \sin\alpha \, d\alpha,$$

$$u_\theta(r,\theta) = \frac{R}{2\pi\mu} \int_0^\pi \sigma_r^0(\alpha) H_\theta\left(\frac{r}{R}, \theta, \alpha\right) \sin\alpha \, d\alpha. \tag{1.27}$$

Here,

$$H_r(r,\theta,\alpha) = \frac{1-2\nu}{1+\nu}\frac{\pi r}{2} + \frac{1-r^2}{2r}\left(2r\frac{\partial U}{\partial r} + U\right)$$

$$+ 2(1-\nu)\frac{1+r^2}{r}U + \frac{1}{r}\,\mathrm{Re}\int_0^1 \left(\frac{Pr^2+Q}{y^{1+n_1}} + \frac{1}{y^2}\right) U(ry)\, dy,$$

$$H_\theta(r,\theta,\alpha) = \frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-r^2)U + \operatorname{Re}\int_0^1 \left(\frac{Sr^2+T}{y^{1+n_1}} + \frac{1}{y^2}\right)U(ry)\,dy\right],$$

$$U(r) \equiv U(r,\theta,\alpha) = \int_0^{\pi/2} d\psi \sum_{n=2}^{\infty} r^n P_n(\lambda) = \frac{K(k)}{h} - \frac{\pi}{2}(1 + r\cos\theta\cos\alpha),$$

$$h^2 = (1-r)^2 + 4r\sin^2\frac{\theta+\alpha}{2}, \qquad k^2 = \frac{4r\sin\theta\sin\alpha}{h^2},$$

and $K(k)$ is a first-order elliptic integral.

Thus, we have obtained the solution of the problem in the theory of elasticity in an explicit form.

We can now go over to a consideration of the problem when the loading (exerted by normal forces only) is not axially symmetric. For this purpose, we turn to the formulas (1.27), putting $\sigma_r^0(\theta) = \delta(\theta)$. In other words, we consider the problem when a concentrated force is applied at the pole. By summing up these solutions over the whole sphere, we can get an integral representation of the solution in the case of an arbitrary loading by normal forces (which may be considered as a sort of Green's function). Since the problem is internal, corrections have to be applied in this method. As a matter of fact, the loading in this case turns out to be unbalanced, and the solution obtained formally in this case is devoid of a physical meaning. It is necessary to apply a compensating load (which is automatically eliminated at the final stage of constructing the solution on account of the condition of self-balancing of external forces). For example, we can apply a compensating concentrated force at the centre. Of course, in this case the solution will have a singularity at the origin of the coordinates, but it disappears upon summation. Another method has been proposed in [3]: the compensating load is taken in the form of a sum of mass forces, distributed uniformly over the volume and directed along the z-axis, and a certain solution compensating the shearing stresses. The solution can then be represented in the form

$$u_r^0 = \left[\cos\theta\,\frac{3r^2}{4R^2}\,\frac{1-2\nu}{1+\nu} + \frac{1}{\pi}H_r\left(\frac{r}{R},\theta,0\right)\right]\frac{P_z}{4\pi\mu R},$$

$$u_\theta^0 = \left[\sin\theta\,\frac{3r^2}{4R^2}\,\frac{1-2\nu}{1+\nu} + \frac{1}{\pi}H_\theta\left(\frac{r}{R},\theta,0\right)\right]\frac{P_z}{4\pi\mu R}. \tag{1.28}$$

Let $\sigma(R,\theta,\varphi) = \sigma(\theta,\varphi)$ be a boundary condition on the surface of a sphere. Writing these formulas for an arbitrary point θ', φ' and adding them, we get

$$u_r(r,\theta',\varphi') = \frac{R}{4\pi^2\mu}\int_0^{2\pi} d\beta \int_0^{\pi} \sigma(\alpha,\beta)H_r\left(\frac{r}{R},\Theta,0\right)\sin\alpha\,d\alpha,$$

$$u_\theta(r,\theta',\varphi') = \frac{R}{4\pi^2\mu}\int_0^{2\pi} d\beta \int_0^{\pi} \sigma(\alpha,\beta) H_\theta\left(\frac{r}{R}, \Theta, 0\right)\sin\alpha\, d\alpha, \tag{1.29}$$

$$\Theta = \arccos\left[\cos\theta' \cos\alpha + \sin\theta' \sin\alpha \cos(\varphi' - \beta)\right].$$

Section 2
Problems of Equilibrium of a Circular Cone and a Wedge

We shall use the method of separation of variables for solving the problems of a circular cone ($0 < r < \infty, 0 \leqslant \theta \leqslant \alpha, -\pi \leqslant \varphi \leqslant \pi$). For this purpose, we shall naturally proceed from the system of spherical coordinates which permits a separation of variables for the system of equations (4.4), Ch. 2, Vol. 1. We start with a quite particular class of solutions, where all displacements are proportional to $1/r$ and, besides, the displacements u_r and u_θ are proportional to $\cos n\varphi$, while u_φ are proportional to $\sin n\varphi$ (n is an arbitrary non-negative integer).

A direct substitution shows that the displacements, which can be represented in the form

$$u_r = \frac{\cos n\varphi}{r}\left[-\frac{\lambda+2\mu}{\mu}\frac{r^2\Delta}{\cos n\varphi} + C\tan^n\frac{\theta}{2} + D\cot^n\frac{\theta}{2}\right],$$

$$u_\theta = \frac{\cos n\varphi}{r\sin\varphi}\left[-\frac{\lambda+3\mu}{2\mu}\sin\theta\frac{d}{d\theta}\left(\frac{r^2\Delta}{\cos n\varphi}\right)\right.$$
$$\left. + \cos\theta\left(C\tan^n\frac{\theta}{2} + D\cot^n\frac{\theta}{2}\right) + G\tan^n\frac{\theta}{2} + H\cot^n\frac{\theta}{2}\right],$$

$$u_\varphi = \frac{\sin n\varphi}{r\sin\theta}\left[n\frac{\lambda+3\mu}{2\mu}\frac{r^2\Delta}{\cos n\varphi}\right. \tag{2.1}$$
$$\left. - \cos\theta\left(C\tan^n\frac{\theta}{2} - D\cot^n\frac{\theta}{2}\right) - G\tan^n\frac{\theta}{2} + H\cot^n\frac{\theta}{2}\right],$$

$$\Delta = \frac{\cos n\varphi}{r^2}\left[A(n+\cos\theta)\tan^n\frac{\theta}{2} + B(n-\cos\theta)\cot^n\frac{\theta}{2}\right],$$

satisfy Lamé's equations. Here, A, B, C, D, G, and H are arbitrary constants.

The cases when $n = 0$ and $n = 1$ must be considered separately. We shall consider five particular solutions [5].

Solution I. Introducing an arbitrary constant P_z, we arrive at the solution

$$u_r = \frac{P_z}{4\pi\mu}\frac{\cos\theta}{r},$$

$$u_\theta = -\frac{\lambda+3\mu}{2(\lambda+2\mu)}\frac{P_z}{4\pi\mu}\frac{\sin\theta}{r},\ u_\varphi = 0,$$

$$\sigma_r = -\frac{3\lambda + 4\mu}{\lambda + 2\mu}\frac{P_z}{4\pi}\frac{\cos\theta}{r^2}, \tag{2.2}$$

$$\sigma_\theta = \sigma_\varphi = \frac{\mu}{\lambda + 2\mu}\frac{P_z}{4\pi}\frac{\cos\theta}{r^2},$$

$$\tau_{r\theta} = \frac{\mu}{\lambda + 2\mu}\frac{P_z}{4\pi}\frac{\sin\theta}{r^2}, \qquad \tau_{\theta\varphi} = \tau_{\varphi r} = 0.$$

It should be observed that this solution is identical to the solution (5.27) and (5.28), Ch. 3, Vol. 1 for a concentrated force in space, if we can transform the latter to a system of spherical coordinates. The force P_z is applied at the origin of the coordinates and is directed along the z-axis.

Solution II. Introducing a constant P_x, we arrive at the solution

$$\begin{aligned}
u_r &= \frac{P_x}{4\pi\mu}\frac{\sin\theta\cos\varphi}{r},\\
u_\theta &= \frac{\lambda + 3\mu}{2(\lambda + 2\mu)}\frac{P_x}{4\pi\mu}\frac{\cos\theta\cos\varphi}{r},\\
u_\varphi &= -\frac{\lambda + 3\mu}{2(\lambda + 2\mu)}\frac{P_x}{4\pi\mu}\frac{\sin\varphi}{r},\\
\sigma_r &= -\frac{3\lambda + 4\mu}{\lambda + 2\mu}\frac{P_x}{4\pi}\frac{\sin\theta\cos\varphi}{r^2},\\
\sigma_\theta &= \sigma_\varphi = \frac{\mu}{\lambda + 2\mu}\frac{P_x}{4\pi}\frac{\sin\theta\cos\varphi}{r^2},\\
\tau_{\theta\varphi} &= 0,\\
\tau_{\varphi r} &= \frac{\mu}{\lambda + 2\mu}\frac{P_x\sin\varphi}{r^2},\\
\tau_{\theta r} &= -\frac{\mu}{\lambda + 2\mu}\frac{P_x}{2\pi}\frac{\cos\theta\cos\varphi}{r^2}.
\end{aligned} \tag{2.3}$$

This solution corresponds to the force P_x applied at the origin of the coordinates along the x-axis.

Solution III. Introducing a constant P_3, we arrive at the solution

$$u_r = \frac{P_3}{r}, \quad u_\theta = -\frac{P_3}{r}\frac{\sin\theta}{1 + \cos\theta}, \quad u_\varphi = 0,$$

$$\sigma_r = -2\mu\frac{P_3}{r^2}, \quad \sigma_\theta = 2\mu\frac{P_3}{r^2}\frac{\cos\theta}{1 + \cos\theta},$$

$$\sigma_\varphi = 2\mu \frac{P_3}{r^2} \frac{1}{1 + \cos\theta}, \qquad (2.4)$$
$$\tau_{\theta\varphi} = \tau_{\varphi r} = 0, \qquad \tau_{r\theta} = 2\mu \frac{P_3}{r^2} \frac{\sin\theta}{1 + \cos\theta}.$$

Solution IV. Introducing a constant P_4, we arrive at the solution

$$u_r = 0, \quad u_\theta = -\frac{P_4}{r} \frac{\cos\theta}{1 + \cos\theta}, \quad u_\varphi = \frac{P_4}{r} \frac{\sin\varphi}{1 + \cos\varphi},$$
$$\sigma_r = 0, \quad \sigma_\theta = -\sigma_\varphi = -2\mu \frac{P_4}{r^2} \frac{(1 - \cos\theta)\cos\varphi}{(1 + \cos\theta)\sin\theta}, \qquad (2.5)$$
$$\tau_{\theta\varphi} = 2\mu \frac{P_4}{r^2} \frac{(1 - \cos\theta)\sin\varphi}{(1 + \cos\theta)\sin\theta},$$
$$\tau_{\varphi r} = -2\mu \frac{P_4}{r^2} \frac{\sin\varphi}{1 + \cos\theta}, \qquad \tau_{r\theta} = 2\mu \frac{P_4}{r^2} \frac{\cos\varphi}{1 + \cos\theta}.$$

Solution V. Introducing a constant P_5, we arrive at the solution

$$u_r = \frac{P_5}{r} \frac{\sin\theta\cos\varphi}{1 + \cos\theta},$$
$$u_\theta = \frac{P_5}{r}\cos\varphi, \qquad u_\varphi = -\frac{P_5}{r}\sin\varphi,$$
$$\sigma_r = -\sigma_\theta = -2\mu \frac{P_5}{r^2} \frac{\sin\theta\cos\varphi}{1 + \cos\theta}, \qquad \sigma_\varphi = 0,$$
$$\tau_{\theta\varphi} = -\mu \frac{P_5}{r^2} \frac{\sin\theta\sin\varphi}{1 + \cos\theta}, \qquad (2.6)$$
$$\tau_{\varphi r} = \mu \frac{P_5}{r^2}\left(2 - \frac{1}{1 + \cos\theta}\right)\sin\varphi,$$
$$\tau_{r\theta} = -\mu \frac{P_5}{r^2}\left(2 - \frac{1}{1 + \cos\theta}\right)\cos\varphi.$$

We add the solutions I and III and require that on the cone $\theta = \alpha$, the boundary values of the stress vector be equal to zero. This condition is automatically satisfied for tangential components. The condition for the normal components leads to the following equation:

$$\frac{\mu}{\lambda + 2\mu} \frac{P_z}{4\pi} + \frac{2\mu P_3}{1 + \cos\alpha} = 0. \qquad (2.7)$$

In this way, we get a solution for a cone whose lateral surface is not subjected to any stress. If we determine the resultant of forces applied to any surface contained within the cone (for a surface of this kind, it is convenient to take a part of a sphere with its centre at the apex), we find that it can be reduced to a force given by

$$F_1 = \frac{P_z}{2(\lambda + 2\mu)} \left[\lambda(1 - \cos^3\alpha) + \mu(1 - \cos\alpha)(1 + \cos^2\alpha)\right] \tag{2.8}$$

and directed along the x-axis into the cone.

Putting $\alpha = \pi/2$ in the above formulas, we arrive at the problem for a half-space (its solution (5.24) and (5.25) in Cartesian coordinates was given in Sec. 6, Ch. 3, Vol. 1).

Combining the solutions II, IV, and V, we can obtain the solution for the case when a concentrated force is applied at the apex of a cone and is directed perpendicular to its axis. The absence of stresses on the surface leads to the following equations:

$$\begin{gathered} 2P_4\frac{1 - \cos\alpha}{\sin\alpha} - P_5\sin\alpha = 0, \\ 2P_4 - P_5(1 + 2\cos\alpha) - \frac{P_x}{4\pi(\lambda + 2\mu)}\cos\alpha(1 + \cos\alpha) = 0, \\ -2P_4\frac{1 - \cos\alpha}{\sin\alpha} + 2P_5\sin\alpha + \frac{P_x}{4\pi(\lambda + 2\mu)}\sin\alpha(1 + \cos\alpha) = 0, \end{gathered} \tag{2.9}$$

whence

$$P_4 = -\frac{P_x(1 + \cos\alpha)^2}{8\pi(\lambda + 2\mu)}, \qquad P_5 = -\frac{P_x(1 + \cos\alpha)}{4\pi(\lambda + 2\mu)}.$$

The direction of the resultant force coincides with the positive direction of the x-axis and is equal to

$$F_2 = \frac{P_x}{4}\frac{(2 + \cos\alpha)\lambda + 2\mu}{\lambda + 2\mu}(1 - \cos\alpha)^2. \tag{2.10}$$

For $\alpha = \pi/2$, we naturally arrive at the problem in which a concentrated tangential force is applied to the boundary point of a half-space.

Solutions similar to those given above have been obtained for the plane problem as well (see, for example, [6]). Let us consider a wedge of an angle 2α ($-\alpha \leqslant \theta \leqslant \alpha$). The solution corresponding to a concentrated force P, applied at the apex and directed along the bisector, is given by

$$\sigma_r = -\frac{2P\cos\theta}{2\alpha + \sin 2\alpha}\frac{1}{r}, \qquad \sigma_\theta = \tau_{r\theta} \equiv 0. \tag{2.11}$$

The solution corresponding to a force applied perpendicular to the bisector is given by

$$\sigma_r = -\frac{2P\sin\theta}{2\alpha - \sin 2\alpha}\frac{1}{r}, \qquad \sigma_\theta = \tau_{r\theta} \equiv 0. \tag{2.12}$$

It is not difficult to obtain a formal solution which can be treated as the solution for the case when a concentrated moment M is applied at the apex:

$$\sigma_r = -\frac{2M \sin 2\theta}{\sin 2\alpha - 2\alpha \cos 2\alpha}\frac{1}{r^2}, \quad \sigma_\theta = 0,$$

$$\tau_{r\theta} = \frac{M(\cos 2\theta - \cos 2\alpha)}{\sin 2\alpha - 2\alpha \cos 2\alpha}\frac{1}{r^2} \tag{2.13}$$

This method suffers from an apparent drawback in that the denominator vanishes for a certain value of α ($\approx 0.714\pi$). A detailed analysis of this solution is given in Sec. 2, Ch. 6.

Section 3
Torsion of Bars with a Polygonal Cross Section

It has been mentioned earlier that the method of separation of variables is found to be effective when the domain is a parallelepiped (or a rectangle) in the corresponding system of curvilinear coordinates. However, it is also possible to apply this method in such cases where the domain is a combination of such domains [7]. We shall describe this method by taking the example of torsion of a prismatic bar with a section as shown in Fig. 31.

It has been pointed out in Sec. 3, Ch. 3, Vol. 1 (formula (3.13)) that the problem of torsion can be reduced to finding a function ψ in the domain D *(ODEMBCO)*, satisfying Poisson's equation $\Delta\psi = -2$ and vanishing at the boundary. We represent the domain D in the form of two overlapping rectangles D_1 *(OABCO)* and D_2 *(ODEFO)*. We shall assume that the problem consists in finding a function ψ_1 in the domain D_1 and a function ψ_2 in the domain D_2, such that these functions are identical over the rectangle D_3 *(OAMFO)* and satisfy Poisson's equation everywhere. Since the funcions ψ_1 and ψ_2 satisfy a second-order equation, it is necessary for their coincidence in the domain D_3 that on the contour of this domain the functions as well as their first derivatives with respect to the normal be identical. Taking this into account, we can write the boundary conditions and the conditions on the segments

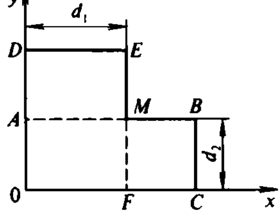

Fig. 31. Polygonal cross section of a bar.

AM and *MF* (we can call these conditions as compatibility conditions) in the following form:

$$\psi_1(0, y) = \psi_1(x, 0) = \psi_1(a, y) = 0,$$

$$\psi_1(x, d_2) = \begin{cases} 0, & x \geqslant d_1, \\ \psi_2(x, d_2), & x < d_1, \end{cases} \tag{3.1}$$

$$\psi_2(x, 0) = \psi_2(0, y) = \psi_2(x, b) = 0,$$

$$\psi_2(d_1, y) = \begin{cases} 0, & y \geqslant d_2, \\ \psi_1(d_1, y), & y < d_2. \end{cases}$$

We shall seek the functions $\psi_1(x, y)$ and $\psi_2(x, y)$ in the form of sums of the functions

$$\psi_{11}(x,y) + \psi_{12}(x,y) \text{ and } \psi_{21}(x,y) + \psi_{22}(x,y). \tag{3.2}$$

The functions ψ_{12} and ψ_{22} are non-zero only in the domain D_3 and satisfy Laplace's equation. The necessary partition of the boundary conditions (3.1) may be carried out in different ways. For the sake of simplification of the subsequent computations, we can write them in the following form:

$$\psi_{11}(0,y) + \psi_{12}(0,y) = \psi_{11}(x,0) = \psi_{11}(a,y) = \psi_{11}(x,d_2) = 0, \tag{3.3}$$

$$\psi_{21}(x,0) + \psi_{22}(x,0) = \psi_{21}(0,y) = \psi_{21}(x,b) = \psi_{21}(d_1,y) = 0, \tag{3.4}$$

$$\psi_{12}(x, 0) = \psi_{12}(d_1, y) = \left(\frac{\partial \psi_{12}}{\partial x}\right)_{x=d_1}$$
$$= \psi_{12}(x, d_2) - \psi_{21}(x, d_2) = 0, \tag{3.5}$$

$$\psi_{22}(0, y) = \psi_{22}(x, d_2) = \left(\frac{\partial \psi_{22}}{\partial y}\right)_{y=d_2}$$
$$= \psi_{22}(d_1,y) - \psi_{11}(d_1,y) = 0. \tag{3.6}$$

One point must be noted here. The vanishing of a harmonic function and its derivative at the boundary segments means that in the domain of harmonicity, the function is identically equal to zero. In view of this, the functions ψ_{12} and ψ_{22} must have singular points in the domain D_3. Hence, while constructing the functions ψ_{ij} in the form of series, we may find that the latter are, generally speaking, divergent. The representations for the functions ψ_1 and ψ_2, however, are found to be convergent. It follows from the conditions (3.3) and (3.4) that the functions ψ_{11} and ψ_{21} may be sought in the form of series

$$\psi_{11}(x, y) = \sum_{k=1}^{\infty} f_k^{(1)}(x) \sin \frac{k\pi y}{d_2},$$

$$\psi_{21}(x, y) = \sum_{k=1}^{\infty} f_k^{(2)}(y) \sin \frac{k\pi x}{d_1}. \tag{3.7}$$

The harmonic functions ψ_{12} and ψ_{22} are chosen in the form of series [8]

$$\psi_{12}(x,y) = \sum_{k=1}^{\infty} v_k^{(1)}(x) \sin \frac{k\pi y}{d_2}, \tag{3.8}$$

$$\psi_{22}(x, y) = \sum_{k=1}^{\infty} v_k^{(2)}(y) \sin \frac{k\pi x}{d_1}. \tag{3.9}$$

In accordance with (3.7), the last two of the conditions (3.5) and (3.6) assume the form

$$\psi_{21}(x, d_2) = \sum_{k=1}^{\infty} f_k^{(2)}(d_2) \sin \frac{k\pi x}{d_1}, \tag{3.10}$$

$$\psi_{22}(d_1, y) = \sum_{k=1}^{\infty} f_k^{(1)}(d_1) \sin \frac{k\pi y}{d_2}. \tag{3.11}$$

The functions $f_k^{(1)}(x)$ and $f_k^{(2)}(y)$ are defined by the expressions

$$f_k^{(1)}(x) = A_k^{(1)} \sinh \frac{k\pi x}{d_2} + B_k^{(1)} \cosh \frac{k\pi x}{d_2} + \frac{4d_2^2}{(k\pi)^3} \left[1 + (-1)^{k+1}\right], \tag{3.12}$$

$$f_k^{(2)}(y) = A_k^{(2)} \sinh \frac{k\pi y}{d_1} + B_k^{(2)} \cosh \frac{k\pi y}{d_1} + \frac{4d_1^2}{(k\pi)^3} \left[1 + (-1)^{k+1}\right]. \tag{3.13}$$

Similarly, for the functions $v_k^{(1)}(x)$ and $v_k^{(2)}(y)$ we get

$$v_k^{(1)}(x) = D_k^{(1)} \sinh \frac{k\pi x}{d_2} + C_k^{(1)} \cosh \frac{k\pi x}{d_2}$$

$$+ (-1)^{k+1} \frac{2k}{\pi} \left(\frac{d_1}{d_2}\right)^2 \sum_{p=1}^{\infty} \frac{f_p^{(2)}(d_2) \sin \frac{p\pi x}{d_1}}{p^2 + \left(\frac{kd_1}{d_2}\right)^2}, \tag{3.14}$$

$$v_k^{(2)}(y) = D_k^{(2)} \sinh \frac{k\pi y}{d_1} + C_k^{(2)} \cosh \frac{k\pi y}{d_1}$$
$$+ (-1)^{k+1} \frac{2k}{\pi} \left(\frac{d_2}{d_1}\right)^2 \sum_{p=1}^{\infty} \frac{f_p^{(1)}(d_1) \sin \dfrac{p\pi y}{d_2}}{p^2 + \left(\dfrac{kd_2}{d_1}\right)^2}. \quad (3.15)$$

In order to determine the constants A_k^i, B_k^i, C_k^i and D_k^i $(i = 1, 2)$, we make use of the equalities

$$\begin{gathered} f_k^{(1)}(a) = f_k^{(2)}(b) = 0, \\ v_k^{(1)}(d_1) = v_k^{(1)\prime}(d_1) = 0, \qquad v_k^{(2)}(d_2) = v_k^{(2)\prime}(d_2) = 0, \\ f_k^{(1)}(0) + v_k^{(1)}(0) = f_k^{(2)}(0) + v_k^{(2)}(0) = 0. \end{gathered} \quad (3.16)$$

The system of equations for these constants may be simplified if we express all the unknowns through $B_k^{(1)}$ and $B_k^{(2)}$. Moreover, this system can be further simplified if we introduce new unknowns $F_k^{(1)}$ and $F_k^{(2)}$ in accordance with the relations

$$\begin{aligned} B_k^{(1)} &= F_k^{(1)} d_1 d_2 \sinh \frac{k\pi d_1}{d_2} \frac{(-1)^{k+1}}{k}, \\ B_k^{(2)} &= F_k^{(2)} d_1 d_2 \sinh \frac{k\pi d_2}{d_1} \frac{(-1)^{k+1}}{k}. \end{aligned} \quad (3.17)$$

Finally, we can write this system as an aggregate of the systems

$$\begin{aligned} F_k^{(1)} &= \sum_{p=1}^{\infty} a_{kp} F_p^{(2)} + \beta_k \\ \hline F_k^{(2)} &= \sum_{p=1}^{\infty} c_{kp} F_p^{(1)} + \gamma_k \end{aligned} \qquad (k = 1, 2, \ldots). \quad (3.18)$$

where

$$a_{kp} = \frac{2kd_1d_2 \sinh \dfrac{p\pi d_2}{d_1} \sinh \dfrac{p\pi}{d_1}(b - d_2) \operatorname{csch} \dfrac{p\pi b}{d_1}}{\pi (p^2 d_2^2 + k^2 d_1^2)},$$

$$c_{kp} = \frac{2kd_1d_2 \sinh \dfrac{p\pi d_1}{d_2} \sinh \dfrac{p\pi}{d_2}(a - d_1) \operatorname{csch} \dfrac{p\pi a}{d_2}}{\pi (p^2 d_1^2 + k^2 d_2^2)},$$

$$\beta_k = -\frac{4d_2}{k^2\pi^3 d_1} \left[1 + (-1)^{k+1}\right] \operatorname{csch} \frac{k\pi d_1}{d_2}$$

$$+ \frac{16k}{\pi^4}\left(\frac{d_1}{d_2}\right)^2 \sum_{p=1,3,\ldots}^{\infty} \frac{1 - \operatorname{csch}\frac{p\pi b}{d_1}\sinh\frac{p\pi d_2}{d_1}}{p^2\left[p^2 + \left(\frac{kd_1}{d_2}\right)^2\right]}, \quad (3.19)$$

$$\gamma_k = -\frac{4d_1}{k^2\pi^3 d_2}\left[1 + (-1)^{k+1}\right]\operatorname{csch}\frac{k\pi d_2}{d_1}$$

$$+ \frac{16k}{\pi^4}\left(\frac{d_2}{d_1}\right)^2 \sum_{p=1,3,\ldots}^{\infty} \frac{1 - \operatorname{csch}\frac{p\pi a}{d_2}\sinh\frac{p\pi d_1}{d_2}}{p^2\left[p^2 + \left(\frac{kd_2}{d_1}\right)^2\right]},$$

The aggregate of the systems (3.18) can be written in the form of a single system if we put

$$F_k^{(1)} = Z_{2k-1}, \quad \beta_k = \alpha_{2k-1}; \quad F_k^{(2)} = Z_{2k}, \quad \gamma_k = \alpha_{2k};$$
$$A_{2k-1,\,2p-1} = A_{2k,\,2p} = 0, \quad A_{2k-1,\,2p} = a_{kp}, \quad A_{2k,\,2p-1} = c_{kp}. \quad (3.20)$$

We shall show that this system is regular (see Sec. 15, Ch. 1, Vol. 1). For this purpose, we consider the equations for odd and even values of m separately. In the first case, we get

$$\sum_{n=1}^{\infty} |A_{mn}| = \sum_{p=1}^{\infty} |a_{kp}| = \frac{2kd_1}{\pi d_2}$$

$$\cdot \sum_{p=1}^{\infty} \frac{\sinh\frac{p\pi d_2}{d_1}\sinh\frac{p\pi}{d_1}(b - d_2)\ \operatorname{csch}\frac{p\pi b}{d_1}}{p^2 + \left(\frac{kd_1}{d_2}\right)^2}$$

$$\leqslant \frac{kd_1}{\pi d_2}\sum_{p=1}^{\infty}\frac{1}{p^2 + \left(\frac{kd_1}{d_2}\right)^2} = \frac{1}{2}\left(\coth\frac{k\pi d_1}{d_2} - \frac{d_2}{k\pi d_1}\right) \leqslant \frac{1}{2}, \quad (3.21)$$

where we have used the inequalities

$$\coth x - \frac{1}{x} \leqslant 1 \quad \text{for} \quad 0 \leqslant x < \infty,$$
$$\sinh x \sinh(y - x)\operatorname{csch} y \leqslant \frac{1}{2}(1 - e^{-2x}) \leqslant \frac{1}{2} \quad \text{for } y > x \geqslant 0 \quad (3.22)$$

and the identity

$$\sum_{p=1}^{\infty} \frac{1}{p^2 + a^2} = \frac{\pi}{2a}\left(\coth a\pi - \frac{1}{a\pi}\right). \tag{3.23}$$

The case of even values of m is considered in a similar way:

$$\sum_{n=1}^{\infty} |A_{mn}| = \sum_{p=1}^{\infty} |c_{kp}|$$

$$= \frac{2kd_2}{\pi d_1} \sum_{p=1}^{\infty} \frac{\sinh \frac{p\pi d_1}{d_2} \sinh \frac{p\pi}{d_2}(a - d_1) \operatorname{csch} \frac{p\pi a}{d_2}}{p^2 + \left(\frac{kd_2}{d_1}\right)^2} \leqslant \frac{1}{2}. \tag{3.24}$$

It should be observed that the free terms in the system of equations are bounded and tend to zero.

Thus, it can be stated that a solution of "truncated" systems leads, in the limit, to the solution of an infinite system.

Section 4
Vibrations of a Plate in the Form of a Sector of a Circle

The problem of determining the natural frequencies of vibrations of a plate can be reduced, in polar system of coordinates, to a consideration of a differential equation of the type

$$\frac{D}{a^4} \Delta^2 w + \gamma\delta \frac{\partial^2 w}{\partial t^2} = 0 \qquad \left(t \geqslant 0, 0 \leqslant \rho = \frac{r}{a} \leqslant 1\right),$$

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}, \qquad D = \frac{E\delta^3}{12(1 - \nu^2)}. \tag{4.1}$$

Here, γ is the density, δ is the thickness of the plate, a is the outer radius, and D is the flexural rigidity.

Let us consider a plate in the form of a sector of a circle with rigidly fastened edges [9]. The solution of Eq. (4.1), satisfying the fastening conditions at the edges $\theta = \pm\theta_0$, is sought in the form of a series

$$w(\rho, \theta, t) = \sum_{n=2}^{\infty} w_n(\rho, t) h_n(\varphi), \qquad \varphi = \frac{\theta}{\theta_0}, \tag{4.2}$$

where $h_n(\varphi)$ are orthonormal Horway polynomials [10, 11], satisfying the following conditions for $\varphi = \pm 1$:

$$h_n(\pm 1) = h_n'(\pm 1) = 0 \qquad \left(h_n' = \frac{dh_n}{d\varphi}\right). \tag{4.3}$$

In order to find the functions $w_n(\rho, t)$, we use the method of reduction to ordinary differential equations [12], along with the approximation that the polynomials $h_n(\varphi)$ have orthogonal first and second derivatives. As a result, we get

$$\frac{D}{a^4\gamma\delta}\left[LLw_n + \frac{1}{\rho^4\theta_0^4}(B_n - A_n^2)w_n\right] + \frac{\partial^2 w_n}{\partial t^2} = 0,$$

$$L = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{A_n}{\rho^2\theta_0^2}, \tag{4.4}$$

$$A_n = \int_{-1}^{+1} (h_n')^2 d\varphi, \qquad B_n = \int_{-1}^{+1} (h_n'')^2 d\varphi.$$

The solution of this equation is obtained with the help of the method of integral transformations with finite limits [13].

We introduce the notation

$$W_n(\gamma_{ni}, t) = \int_0^1 \rho w_n(\rho, t) G_n(\rho\gamma_{ni})\, d\rho, \tag{4.5}$$

where $G_n(\rho\gamma_{ni})$ is the kernel of the integral transformation, which is yet to be determined, and γ_{ni} is a certain parameter.

Integrating by parts successively, we can show that

$$\int_0^1 \rho\left[LLw_n + \frac{1}{\rho^4\theta_0^4}(B_n - A_n^2)w_n\right] G(\rho\gamma_{ni})\, d\rho$$

$$= \left[K(\rho\gamma_{ni}, t)\right]_{\rho=0}^{\rho=1} + \int_0^1 \rho w_n \left[LLG_n + \frac{1}{\rho^4\theta_0^4}(B_n - A_n^2)G_n\right] d\rho. \tag{4.6}$$

Here,

$$K(\rho\gamma_{ni}, t) = \rho\left\{\frac{\partial^3 w_n}{\partial\rho^3}G_n + \frac{\partial^2 w_n}{\partial\rho^2}\left(\frac{G_n}{\rho} - \frac{dG_n}{d\rho}\right)\right.$$
$$+ \frac{\partial w_n}{\partial\rho}\left[\frac{d^2 G_n}{d\rho^2} - \frac{1}{\rho^2}\left(1 - \frac{A_n}{\theta_0^2}\right)G_n\right]$$
$$\left. - w_n\left[\frac{d^3 G_n}{d\rho^3} + \frac{1}{\rho}\frac{d^2 G_n}{d\rho^2} - \frac{1}{\rho^2}\left(1 + \frac{A_n}{\theta_0^2}\right)G_n\right]\right\}. \tag{4.7}$$

Suppose that the function $G_n(\rho\gamma_{ni})$ satisfies the equation

$$LLG_n + \frac{1}{\rho^4\theta_0^4}(B_n - A_n^2)G_n = \gamma_{ni}G_n \tag{4.8}$$

and the boundary conditions

$$G_n = \frac{dG_n}{d\rho} = 0 \qquad (\rho = 0,\ \rho = 1). \tag{4.9}$$

We multiply (4.4) by ρG_n and integrate over ρ ($0 \leqslant \rho \leqslant 1$). Taking into account Eqs. (4.5), (4.8), and (4.9), as well as the conditions

$$w_n = \frac{dw_n}{d\rho} = 0 \qquad (\rho = 0, \rho = 1),$$

we obtain the following equation:

$$\frac{d^2 W_n}{dt^2} + \omega_{ni}^2 W_n = 0, \qquad \omega_{ni}^2 = \frac{\gamma_{ni} D}{a^4 \gamma \delta}. \tag{4.10}$$

Let us now determine the eigenvalues γ_{ni} and the eigenfunctions $G_n(\rho\gamma_{ni})$ of the boundary value problem (4.8), (4.9). First of all, we transform Eq. (4.8) with the help of the substitution of the variable $\rho = e^x$. This gives

$$L_1 L_2 G_n - \gamma_{ni} e^{4x} G_n = 0, \qquad L_{1,2} = \frac{d^2}{dx^2} - 2\frac{d}{dx} - \varkappa_{1,2}. \tag{4.11}$$

Here, $\varkappa_{1,2}$ are the roots of the equation

$$\varkappa^2 + \frac{2A_n}{\theta_0^2}\varkappa + \frac{1}{\theta_0^4}(B_n - 4A_4\theta_0^2) = 0. \tag{4.12}$$

The solution of (4.11) is sought in the form of a series

$$G_n = \sum_{m=0}^{\infty} \gamma_{ni}^m G_n^{(m)}. \tag{4.13}$$

Substituting (4.13) into (4.11) and equating the coefficients of like powers of γ_{ni}, we get a sequence of equations with constant coefficients

$$L_1 L_2 G_n^{(0)} = 0, \qquad L_1 L_2 G_n^{(m)} = e^{4x} G_n^{(m-1)} \qquad (m = 1, 2, \dots). \tag{4.14}$$

Henceforth, we shall be considering plates in the form of a sector, for which the apex angle satisfies the condition

$$2\theta_0 < \left(\frac{B_n - A_n^2}{A_n}\right)^{1/2} < \left(\frac{B_2 - A_2^2}{A_2}\right)^{1/2} = 2.74 \qquad (n = 3, 4, \dots). \tag{4.15}$$

In this case, the characteristic polynomial of the operator $L_1 L_2$,

$$P(\lambda) \equiv \left[\lambda(\lambda - 2) - \frac{A_n}{\theta_0^2}\right]^2 + \frac{1}{\theta_0^4}(B_4 - 4A_n\theta_0^2 - A_n^2), \tag{4.16}$$

has complex-conjugate roots of the form

$$\begin{aligned} \lambda_{1n}^0 &= \lambda_{1n} + i\lambda_n, \qquad \lambda_{2n}^0 = \lambda_{2n} + i\lambda_n, \\ \lambda_{3n}^0 &= \lambda_{1n} - i\lambda_n, \qquad \lambda_{4n}^0 = \lambda_{2n} - i\lambda_n, \end{aligned} \tag{4.17}$$

where

$$\lambda_{1n} = 1 + \frac{1}{\sqrt{2}}\left[\left(1 - \frac{2A_n}{\theta_0^2} + \frac{B_n}{\theta_0^4}\right)^{1/2} + \frac{A_n}{\theta_0^2} + 1\right]^{1/2},$$

$$\lambda_n = \frac{1}{\sqrt{2}}\left[\left(1 - \frac{2A_n}{\theta_0^2} + \frac{B_n}{\theta_0^4}\right)^{1/2} - \frac{A_n}{\theta_0^2} - 1\right]^{1/2},$$

$$\lambda_{2n} = 2 - \lambda_{1n}.$$

Solving the system (4.14) successively, we obtain a general solution of Eq. (4.11):

$$G_n(\rho\gamma_{ni}) = c_1 \mathrm{Re} Z_n^{(1)}(\rho\gamma_{ni}) + c_2 \mathrm{Im} Z_n^{(1)}(\rho\gamma_{ni}) + c_3 \mathrm{Re} Z_n^{(2)}(\rho\gamma_{ni}) + c_4 \mathrm{Im} Z_n^{(2)}(\rho\gamma_{ni}). \tag{4.18}$$

Here,

$$Z_n^{(1)}(\rho\gamma_{ni}) = \rho^{\lambda_{1n}^0}\left\{1 + \sum_{m=1}^{\infty}(\rho^4\gamma_{ni})^m\left[\prod_{j=1}^{m} P(4j + \lambda_{1n}^0)\right]^{-1}\right\},$$

$$Z_n^{(2)}(\rho\gamma_{ni}) = \rho^{\lambda_{2n}^0}\left\{1 + \sum_{m=1}^{\infty}(\rho^4\gamma_{ni})^m\left[\prod_{j=1}^{m} P(4j + \lambda_{2n}^0)\right]^{-1}\right\}. \tag{4.19}$$

Satisfying the conditions of boundedness of the function $G_n(\rho\gamma_{ni})$, for $\rho = 0$, we put $c_3 = c_4 = 0$. Then the transformation kernel can be written in the form:

$$G_n(\rho\gamma_{ni}) = c_1 \mathrm{Re} Z_n^{(1)}(\rho\gamma_{ni}) + c_2 \mathrm{Im} Z_n^{(1)}(\rho\gamma_{ni}). \tag{4.20}$$

The function $Z_n^{(1)}(\rho\gamma_{ni})$ may be represented in a different form if we take into account the following dependence:

$$P(4j + \lambda_{1n}^0) = 256j(j - 1 + \beta_{1n}^0)(j - 1 + \beta_{2n}^0)(j - 1 + \beta_{3n}^0),$$

$$\prod_{j=1}^{m} P(4j + \lambda_{1n}^0) = (256)^m m!(\beta_{1n}^0, m)(\beta_{2n}^0, m)(\beta_{3n}^0, m). \tag{4.21}$$

The following notation has been used here:

$$(\beta, m) = \beta(\beta + 1) \dots (m - 1 + \beta),$$

$$\beta_{1n}^0 = \frac{1}{2}(1 + \lambda_{1n}^0), \quad \beta_{2n}^0 = \frac{1}{2}(1 + \lambda_{1n}), \quad \beta_{3n}^0 = \frac{1}{2}(2 + i\lambda_n).$$

The first of the formulas (4.19) can then be written in the following form:

$$Z_n^{(1)}(\rho\gamma_{ni}) = \rho^{\lambda_{1n}^0}\sum_{m=0}^{\infty}\left(\frac{\gamma_{ni}\rho^4}{256}\right)^m \frac{1}{(\beta_{1n}^0, m)(\beta_{2n}^0, m)(\beta_{3n}^0, m)m!} \tag{4.22}$$

It is well known [4] that the series in this formula is a generalized hypergeometric function of the type $F_3(\beta_{1n}^0, \beta_{2n}^0, \beta_{3n}^0; \gamma_{ni}\rho^4/256)$ and converges for all finite values of $\gamma_{ni}\rho^4/256$.

With the help of the conditions (4.9), we get the following system for determining the constants c_1 and c_2:

$$c_1 \mathrm{Re} Z_n^{(1)}(\gamma_{ni}) + c_2 \mathrm{Im} Z_n^{(1)}(\gamma_{ni}) = 0,$$

$$c_1 \mathrm{Re}\left[\frac{d}{d\rho} Z_n^{(1)}(\gamma_{ni}\rho)\right]_{\rho=1} + c_2 \mathrm{Im}\left[\frac{d}{d\rho} Z_n^{(1)}(\gamma_{ni}\rho)\right]_{\rho=1} = 0. \tag{4.23}$$

Equating the determinant of this system to zero, we get an equation for finding the eigenvalues γ_{ni}:

$$\Big\{\mathrm{Re}\left[Z_n^{(1)}(\gamma_{ni}\rho)\right]\mathrm{Im}\left[\frac{d}{d\rho}Z_n^{(1)}(\gamma_{ni}\rho)\right] - \mathrm{Im}\left[Z_n^{(1)}(\gamma_{ni}\rho)\right]\mathrm{Re}\left[\frac{d}{d\rho}Z_n^{(1)}(\gamma_{ni}\rho)\right]\Big\}_{\rho=1} = 0. \tag{4.24}$$

With the help of (4.24), we can write the final expression for the transformation kernel in the following form [assuming, for example, that one of the constants is equal to $\mathrm{Im}\, Z_n^{(1)}(\gamma_{ni}\rho)$]:

$$G_n(\rho\gamma_{ni}) = \mathrm{Re}\,[Z_n^{(1)}(\gamma_{ni}\rho)]\,\mathrm{Im}\,[Z_n^{(1)}(\gamma_{ni}\rho)] - \mathrm{Re}\,[Z_n^{(1)}(\gamma_{ni}\rho)]\,\mathrm{Im}\,[Z_n^{(1)}(\gamma_{ni}\rho)]. \tag{4.25}$$

In order to determine the asymptotic value of the roots γ_{ni}, we consider Eq. (4.8) for large values of γ_{ni}, so that the second term on the left-hand side of the equation can be neglected. In this case, we proceed from the approximate equation for the function G_n:

$$LLG_n - \gamma_{ni} G_n = 0. \tag{4.26}$$

The bounded solution of (4.26) for $\rho = 0$ has the form

$$G_n = c_1 I_\nu(\omega\rho) + c_2 J_\nu(\omega\rho), \tag{4.27}$$

where $\nu = A_n^{1/2}/\theta^2$, $\omega = \gamma_{ni}^{1/4}$, and J_ν and I_ν are the ν-th order Bessel functions for the real and imaginary argument.

Satisfying the conditions (4.9) and using the properties of Bessel functions, we get the following equation for determining γ_{ni}:

$$J_\nu(\omega)J_{\nu+1}(\omega) + I_{\nu+1}(\omega)J_\nu(\omega) = 0. \tag{4.28}$$

Substituting the corresponding asymptotic formulas for Bessel functions into (4.28), we get the following approximate expression:

$$\sin\left(\omega - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) + \cos\left(\omega - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) = 0. \tag{4.29}$$

It follows from this equation that

$$\omega = \gamma_{ni}^{1/4} = m\pi + \nu\frac{\pi}{2}. \tag{4.30}$$

Calculations were made for the corresponding values of γ_{ni} by using Eq. (4.24) for plates in the form of a quarter of a circle, fastened along the entire contour. The values $\gamma_{ni}/256$ for the case of vibrations symmetric about the axis $\theta = 0$ are given in Table 4.

TABLE 4

	1	2	3	4	5	6	7
2	9.328	43.253	128.435	299.89	601.881	1088.27	1757.64
	(7.437)	(35.262)	(107.804)	(256.96)	(527.77)	(968.77)	(1640.13)
4	77.256	216.282	463.511	922.113	1780.3	2850.52	4112.81
	(105.24)	(253.02)	(519.3)	(955.03)	(1620.3)	(2584.2)	(3925.2)
6	334.241	761.42	1432.12	2421.06	3842.71	5164.57	8851.93
	(532.57)	(975.95)	(1651.35)	(2628.3)	(3985.5)	(5810.8)	(8201.1)

For the sake of comparison, the values of $\gamma_{ni}/256$ computed with the help of the formula (4.30) have been given in each column (in brackets). It is clear from these results that for $i = 7$, the error in the value of γ_{ni}, introduced by using the formula (4.30), does not exceed 8.0%.

Section 5
On the Steady-State Vibrations in a Plane with a Cut

We shall consider a dynamic problem in the theory of elasticity for the case of a plane with a cut, whose solution can be obtained by using the method of separation of variables [14].

Consider a steady-state wave process, assuming that the time dependence of all quantities is expressed by the factor $\exp(-i\omega t)$, where ω is the frequency of vibrations and t is the time.

We assume that a normal load $q(\theta)e^{-i\omega t}$ is applied to the sides of a cut, and introduce a system of elliptical coordinates

$$x + iy = l\cosh(\rho + i\theta).$$

The contour of the cut is defined by $\rho = 0$ $(0 \leqslant \theta \leqslant 2\pi)$.

The equations of motion and the elasticity equations for the amplitude values of stresses and displacements in the system of coordinates chosen by us are given by

$$\begin{aligned} &\frac{\partial}{\partial\rho}(H\sigma_\rho) + \frac{\partial}{\partial\theta}(H\sigma_{\rho\theta}) + \frac{\partial H}{\partial\theta}\tau_{\rho\theta} - \frac{\partial H}{\partial\rho}\sigma_\theta = -H^2\rho_0\omega^2 u_\rho, \\ &\frac{\partial}{\partial\theta}(H\sigma_\theta) + \frac{\partial}{\partial\rho}(H\tau_{\rho\theta}) + \frac{\partial H}{\partial\rho}\tau_{\rho\theta} - \frac{\partial H}{\partial\theta}\sigma_\rho = -H^2\rho_0\omega^2 u_\theta; \end{aligned} \tag{5.1}$$

$$\sigma_\rho = 2\mu\left(\frac{1}{H}\frac{\partial u_\rho}{\partial\rho} + \frac{1}{H^2}\frac{\partial H}{\partial\theta}u_\theta\right) + \frac{\lambda}{H^2}\left[\frac{\partial}{\partial\rho}(Hu_\rho) + \frac{\partial}{\partial\theta}(Hu_\theta)\right],$$

$$\tau_{\rho\theta} = \mu\left[\frac{\partial}{\partial\rho}\left(\frac{u_\theta}{H}\right) + \frac{\partial}{\partial\theta}\left(\frac{u_\rho}{H}\right)\right], \tag{5.2}$$

$$\sigma_\theta = 2\mu\left(\frac{1}{H}\frac{\partial u_\theta}{\partial\theta} + \frac{1}{H^2}\frac{\partial H}{\partial\rho}u_\rho\right) + \frac{\lambda}{H^2}\left[\frac{\partial}{\partial\rho}(Hu_\rho) + \frac{\partial}{\partial\theta}(Hu_\theta)\right].$$

Here, $H^2 = (1/2)l^2(\cosh 2\rho - \cos 2\theta)$, ρ_0 is the density of the material, and ω is the frequency.

We express the displacement and stress components in the usual way through two functions (potentials) φ and ψ (see Sec. 5, Ch. 3, Vol. 1):

$$u_\rho = \frac{1}{H}\frac{\partial\varphi}{\partial\rho} + \frac{1}{H}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{1}{H}\frac{\partial\varphi}{\partial\theta} - \frac{1}{H}\frac{\partial\psi}{\partial\rho}.$$

Here, φ and ψ are the solutions of the equations

$$\begin{gathered} \nabla^2\varphi + 2k_1(\cosh 2\rho - \cos 2\theta)\varphi = 0 \\ \nabla^2\psi + 2k_2(\cosh 2\rho - \cos 2\theta)\psi = 0. \end{gathered} \qquad \left(\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}\right). \tag{5.3}$$

In these equations, $k_1 = \omega^2 l^2/(4a^2)$ and $k_2 = \omega^2 l^2/(4b^2)$ are dimensionless quantities, while a and b are the velocities of the longitudinal and transverse waves.

In such a representation the formulas (5.2) assume the form

$$\begin{aligned} \frac{1}{2\mu}\tau_{\rho\theta} &= \frac{1}{H}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial\varphi}{\partial\rho} + \frac{1}{H}\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{H^3}\frac{\partial H}{\partial\rho}\left(\frac{\partial\varphi}{\partial\rho} - \frac{\partial\psi}{\partial\rho}\right) + \frac{\omega^2}{2b^2}\psi, \\ \frac{1}{2\mu}\sigma_\rho &= -\frac{1}{H}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial\varphi}{\partial\theta} - \frac{1}{H}\frac{\partial\psi}{\partial\rho}\right) - \frac{1}{H^3}\frac{\partial H}{\partial\rho}\left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right) + \frac{\omega^2}{2b^2}\varphi, \\ \frac{1}{2\mu}\sigma_\theta &= \frac{1}{H}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial\varphi}{\partial\theta} - \frac{1}{H}\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{H^3}\frac{\partial H}{\partial\rho}\left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right) - \frac{\lambda}{2\mu}\frac{\omega^2}{a^2}\varphi. \end{aligned} \tag{5.4}$$

Separation of variables in Eqs. (5.3) leads to ordinary differential equations whose solutions are Mathieu functions [15].

Suppose that stresses $\sigma_\rho^{(0)}$ and $\tau_{\rho\theta}^{(0)} = 0$ are given on the contour of the slit $\rho = 0$ $(-\pi \leqslant \theta \leqslant \pi)$. Then, taking into account the first two of the equalities (5.4) and considering that $\left.\frac{\partial H}{\partial\rho}\right|_{\rho=0} = 0$, $\left.H\right|_{\rho=0} = l\sin\theta$, we get

$$\begin{aligned} \left.\left(\frac{\partial\varphi}{\partial\theta} - \frac{\partial\psi}{\partial\rho}\right)\right|_{\rho=0} &= -2k_2\sin\theta\int_{\pi/2}^{\theta}\sin\theta\varphi\Big|_{\rho=0}d\theta - \frac{l^2}{2\mu}\sin\theta\int_{\pi/2}^{\theta}\sin\theta\sigma_\rho^{(0)}\Big|_{\rho=0}d\theta, \\ \left.\left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right)\right|_{\rho=0} &= -2k_2\sin\theta\int_0^{\theta}\sin\theta\cdot\psi\Big|_{\rho=0}d\theta. \end{aligned} \tag{5.5}$$

The relations (5.5) have been obtained by taking into account the following obvious expressions:

$$u_\rho|_{\rho=0,\,\theta=0} = 0, \qquad u_\theta|_{\rho=0,\,\theta=\pi/2} = 0.$$

By using the symmetry of the state of stress and taking into consideration the conditions at infinity, we can represent the solutions of Eqs. (5.3) in the form

$$\varphi(\rho,\theta) = \sum_{m=0}^{\infty} C_m \operatorname{Fek}_{2m}(\rho, k_1)\operatorname{ce}_{2m}(\theta, k_1),$$
$$\psi(\rho,\theta) = \sum_{m=0}^{\infty} D_m \operatorname{Gek}_{2m}(\rho, k_2)\operatorname{se}_{2m+2}(\theta, k_2). \tag{5.6}$$

Here, C_m and D_m are constants, $\operatorname{ce}_{2m}(\theta, k_1)$ and $\operatorname{se}_{2m+2}(\theta, k_2)$ are periodic Mathieu solutions, and $\operatorname{Fek}_{2m}(\rho, k_1)$ and $\operatorname{Gek}_{2m}(\rho, k_2)$ are the second solutions of the Mathieu equation.

The periodic Mathieu functions may be represented in the form of series

$$\operatorname{ce}_{2m}(\theta, k_1) = \sum_{r=0}^{\infty} A_{2r}^{(2m)} \cos\ 2r\theta,$$

$$\operatorname{se}_{2m+2}(\theta, k_2) = \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \sin\ (2r+2)\theta,$$

where $A_{2r}^{(2m)}$ and $B_{2r+2}^{(2m+2)}$ are functions of k_1 and k_2. The method of their derivation is given in [15].

Consider the behaviour of the functions $\operatorname{Fek}_{2m}(\rho, k_1)$ and $\operatorname{Gek}_{2m+2}(\rho, k_2)$ for large values of ρ. As $\rho \to \infty$, we get ($H_{2m}^{(1)}(x)$ is the Hankel function)

$$\operatorname{Fek}_{2m}(\rho, k_1) \to \frac{i}{2}\,(-1)^m p_{2m} H_{2m}^{(1)}\left(\frac{2}{l}\sqrt{k_1}\,r\right),\quad \frac{1}{2}\,le^\rho \to r,$$
$$\operatorname{Gek}_{2m+2}(\rho, k_2) \to \frac{i}{2}\,(-1)^{m+1} s_{2m+2} H_{2m+2}^{(1)}\left(\frac{2}{l}\sqrt{k_2}\,r\right); \tag{5.7}$$

$$p_{2m} = \frac{\operatorname{ce}_{2m}(0, k_1)\operatorname{ce}_{2m}\left(\frac{\pi}{2}, k_1\right)}{A_0^{(2m)}},$$

$$s_{2m+2} = \frac{se'_{2m+2}(0, k_2)se'_{2m+2}\left(\frac{\pi}{2}, k_2\right)}{k_2 B_2^{(2m+2)}}. \tag{5.8}$$

If we take into account Eqs. (5.7), we find that the potentials φ and ψ, which are defined by the series (5.6), satisfy the radiation conditions (4.38), Ch. 3, Vol. 1, in a somewhat modified form:

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial\varphi}{\partial r} - i\frac{2\sqrt{k_1}}{l}\varphi\right) = 0, \qquad \varphi = O\left(\frac{1}{\sqrt{r}}\right),$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial\psi}{\partial r} - i\frac{2\sqrt{k_2}}{l}\psi\right) = 0, \qquad \psi = O\left(\frac{1}{\sqrt{r}}\right).$$

In the general case, the functions $\mathrm{Fek}_{2m}(\rho, k_1)$ and $\mathrm{Gek}_{2m+2}(\rho, k_2)$ are complex functions and may be represented in the following form:

$$\mathrm{Fek}_{2m}(\rho, k_1) = -\frac{1}{2}\mathrm{Fey}_{2m}(\rho, k_1) + \frac{i}{2}\mathrm{Ce}_{2m}(\rho, k_1),$$

$$\mathrm{Gek}_{2m+2}(\rho, k_2) = -\frac{1}{2}\mathrm{Gey}_{2m+2}(\rho, k_2) + \frac{i}{2}\mathrm{Se}_{2m+2}(\rho, k_2), \tag{5.9}$$

$$\mathrm{Ce}_{2m}(\rho, k_1) = \sum_{r=0}^{\infty} A_{2r}^{(2m)} \cosh 2r\rho,$$

$$\mathrm{Se}_{2m+2}(\rho, k_2) = \sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} \sinh (2r+2)\rho;$$

$$\mathrm{Fey}_{2m}(\rho, k_1) = \frac{ce_{2m}\left(\frac{\pi}{2}, k_1\right)}{A_0^{(2m)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2m)} Y_{2r}(2k_1 \cosh \rho),$$

$$\mathrm{Gey}_{2m+2}(\rho, k_2) \tag{5.10}$$

$$= -\frac{s_{2m+2}}{B_2^{(2m+2)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2m+2)} \Big[J_r(e^{-\rho}\sqrt{k_2})Y_{r+2}(e^{\rho}\sqrt{k_2}) - J_{r+2}(e^{-\rho}\sqrt{k_2})Y_r(e^{\rho}\sqrt{k_2})\Big].$$

Here, $\mathrm{Ce}_{2m}(\rho, k_1)$ and $\mathrm{Se}_{2m+2}(\rho, k_2)$ are modified Mathieu functions of the first kind, while J_{2r} and Y_{2r} are the Bessel functions of the first and the second kind respectively.

Substituting (5.6) into the boundary conditions (5.5), we get

$$\sum_{m=0}^{\infty} C_m \mathrm{Fek}_{2m}(0, k_1)\mathrm{ce}'_{2m}(\theta, k_1) - \sum_{n=0}^{\infty} D_n \mathrm{Gek}'_{2n+2}(0, k_2)\mathrm{se}'_{2n+2}(\theta, k_2)$$
$$= -2k_2 \sum_{m=0}^{\infty} C_m \mathrm{Fek}'_{2m}(0, k_1)\left[\sin\theta \int_{\pi/2}^{\theta} \sin\theta\, \mathrm{ce}_{2m}(\theta, k_1)\, d\theta\right] - \frac{l^2}{2\mu}\sin\theta \int_{\pi/2}^{\theta} \sin\theta \sigma_\rho\Big|_{\rho=0} d\theta; \tag{5.11}$$

$$\sum_{n=0}^{\infty} C_n \mathrm{Fek}'_{2n}(0, k_1)\mathrm{ce}_{2n}(\theta, k_1) + \sum_{m=0}^{\infty} D_m \mathrm{Gek}_{2m+2}(0, k_2)\mathrm{se}_{2m+2}(\theta, k_2)$$
$$= -2k_2 \sum_{m=0}^{\infty} D_m \mathrm{Gek}_{2m+2}(0, k_2)\left[\sin\theta \int_0^{\theta} \sin\theta\, \mathrm{se}'_{2m+2}(\theta, k_2)\, d\theta\right]. \tag{5.12}$$

We multiply Eq. (5.11) by $\mathrm{se}_{2n+2}(\theta, k_2)$, Eq. (5.12) by $\mathrm{ce}_{2n}(\theta, k_1)$, and integrate with respect to θ between the limits 0 to 2π. Taking into account the orthogonality of periodic Mathieu solutions, we get two infinite systems of equations for determining C_m and D_m:

$$C_n \mathrm{Fek}_{2n}(0, k_1) + \sum_{m=0}^{\infty} D_m \mathrm{Gek}_{2m+2}(0, k_2)\alpha_{nm} = 0, \tag{5.13}$$

$$\sum_{m=0}^{\infty} C_m \mathrm{Fek}_{2m}(0, k_1)\beta_{nm} - D_n \mathrm{Gek}'_{2n+2}(0, k_2) = -\frac{l^2}{2\mu} \sum_{m=0}^{\infty} f_{nm} q_m. \tag{5.14}$$

Here, q_m are the coefficients of expansion of normal stresses on the contour into a series in even Mathieu functions:

$$q_m = \frac{1}{\pi} \int_0^{2\pi} \sigma_\rho\Big|_{\rho=0} \mathrm{ce}_{2m}(\theta, k_1)\, d\theta, \quad \sigma_\rho\Big|_{\rho=0} = \sum_{m=0}^{\infty} q_m \mathrm{ce}_{2m}(\theta, k_1),$$

$$\alpha_{nm} = \frac{1}{\pi} \int_0^{2\pi} \mathrm{se}'_{2m+2}(\theta, k_2)\mathrm{ce}_{2n}(\theta, k_1)d\theta$$

$$+ \frac{2k_2}{\pi} \int_0^{2\pi} \left[\sin\theta\,\mathrm{ce}_{2n}(\theta, k_1) \int_0^{\theta} \sin\theta_1 \mathrm{se}_{2m+2}(\theta_1, k_2)d\theta_1\right]d\theta, \tag{5.15}$$

$$f_{nm} = \frac{1}{\pi} \int_0^{2\pi} \left[\sin\theta\ \mathrm{se}_{2n+2}(\theta, k_2) \int_{\pi/2}^{\theta} \sin\theta_1 \mathrm{ce}_{2m}(\theta_1, k_1)d\theta_1\right]d\theta,$$

$$\beta_{nm} = \frac{1}{\pi} \int_0^{2\pi} \mathrm{ce}'_{2m}(\theta, k_1)\mathrm{se}_{2n+2}(\theta, k_2)d\theta$$

$$+ \frac{2k_2}{\pi} \int_0^{2\pi} \left[\sin\theta\ \mathrm{se}_{2n+2}(\theta, k_2) \int_{\pi/2}^{\theta} \sin\theta_1\ \mathrm{ce}_{2m}(\theta_1, k_1)d\theta_1\right]d\theta.$$

Thus, α_{nm}, β_{nm}, and f_{nm} are the coefficients of expansion in the following series:

$$\mathrm{ce}'_{2m}(\theta, k_1) + 2k_2 \sin\theta \int_{\pi/2}^{\theta} \sin\theta\ \mathrm{ce}_{2m}(\theta, k_1)d\theta = \sum_{n=0}^{\infty} \beta_{nm}\ \mathrm{se}_{2n+2}(\theta, k_2),$$

$$\mathrm{se}'_{2m+2}(\theta, k_2) + 2k_2 \sin\theta \int_0^{\theta} \sin\theta\ \mathrm{se}_{2m+2}(\theta, k_2)d\theta = \sum_{n=0}^{\infty} \alpha_{nm}\ \mathrm{ce}_{2n}(\theta, k_1),$$

$$\sin\theta \int_{\pi/2}^{\theta} \sin\theta\ \mathrm{ce}_{2m}(\theta, k_1)d\theta = \sum_{n=0}^{\infty} f_{nm}\ \mathrm{se}_{2n+2}(\theta, k_2).$$

Using the expansions of the functions $\mathrm{ce}_{2n}(\theta, k_1)$ and $\mathrm{se}_{2n+2}(\theta, k_2)$ into Fourier series, we can show that

$$\alpha_{nm} = \sum_{r=0}^{\infty} 2rA_{2r}^{(2n)}B_{2r}^{(2m+2)}$$

$$+ \frac{k_2}{2} \sum_{r=0}^{\infty} \frac{B_{2r+2}^{(2m+2)}}{(2r+1)(2r+3)} [(2r+3)A_{2r}^{(2n)} + (2r+1)A_{2r+3}^{(2n)} - 4(r+1)A_{2r+2}^{(2n)}],$$

$$\beta_{nm} = -\alpha_{nm},$$

$$f_{nm} = -\frac{1}{2} A_0^{(2m)} B_2^{(2n+2)}$$

$$+\frac{1}{4} \sum_{r=1}^{\infty} \frac{A_{2r}^{(2m)}}{4r^2-1} [4rB_{2r}^{(2n+2)} - (2r-1)B_{2r+2}^{(2n+2)} - (2r+1)B_{2r-2}^{(2n+2)}].$$

Hence, for a given value of the normal load on the contour of a cut, our problem is reduced to the determination of the constants C_n and D_n $(n = 0, 1, 2, \ldots)$ from two infinite systems (5.13) and (5.14).

Let us determine the normal stresses for points of the real axis on the extension of the cut:

$$\sigma_\theta\Big|_{\theta=0} = \frac{2\mu}{l^2 \sinh^3\rho} \left[\sinh\rho \left(\frac{\partial^2\varphi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial\rho\,\partial\theta}\right)_{\theta=0}\right.$$

$$\left. + \cosh\rho \left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right)_{\theta=0}\right] - \frac{\lambda\rho\omega^2\varphi}{\lambda+2\mu}\Big|_{\theta=0} \,. \qquad (5.16)$$

Then, taking (5.16) into account, we find that the stress intensity factor K is equal to $(\sqrt{s} = \sqrt{2l}\sinh \rho/2)$

$$K = \lim_{\rho\to 0} \left[\sqrt{2\pi s}\,\sigma_\theta\Big|_{\theta=0}\right] = \frac{\mu\sqrt{\pi l}}{l^2} \lim_{\rho\to 0} \left(\frac{\partial^3\varphi}{\partial\rho\partial\theta^2} - \frac{\partial^3\psi}{\partial\rho^2\partial\theta}\right)_{\theta=0}.$$

Substituting the expressions for φ and ψ and passing to the limit, we get

$$K = \frac{\mu\sqrt{\pi l}}{l^2} \sum_{m=0}^{\infty} [C_m \,\mathrm{Fek}'_{2m}(0, k_1)\mathrm{ce}''_{2m}(0, k_1)$$

$$- D_m \mathrm{Gek}''_{2m+2}(0, k_2)\mathrm{se}'_{2m+2}(0, k_2)]. \qquad (5.17)$$

Taking into account the relations

$$\mathrm{ce}''_{2m}(0, k_1) = (2k_1 - a_{2m})\mathrm{ce}_{2m}(0, k_1),$$

$$\mathrm{Gek}''_{2m+2}(0, k_2) = (b_{2m+2} - 2k_2)\mathrm{Gek}_{2m+2}(0, k_2),$$

the expression (5.17) may be written in the form

$$K = \frac{\mu\sqrt{\pi l}}{l^2} \sum_{m=0}^{\infty} [C_m \mathrm{Fek}'_{2m}(0, k_1)(2k_1 - a_{2m})\mathrm{ce}_{2m}(0, k_1)$$

$$- D_m \mathrm{Gek}_{2m+2}(0, k_2)(b_{2m+2} - 2k_2)\mathrm{se}'_{2m+2}(0, k_2)]. \qquad (5.18)$$

Here, a_{2m} and b_{2m+2} are the eigenvalues of the Mathieu functions $\mathrm{ce}_{2m}(\theta, k_1)$ and $\mathrm{se}_{2m+2}(\theta, k_2)$ respectively.

While calculating, for example, the stress intensity factor in accordance with (5.18), the main problem lies in solving the systems (5.13) and (5.14). However, the solution is simplified for small values of k_1 and k_2.

For real materials, the values of the arguments k_1 and k_2 of the Mathieu functions, in terms of which the solution is represented, are given by

$$k_1 = \frac{\omega^2 l^2 \rho_0}{4(\lambda + 2\mu)} \sim 0.74 \cdot 10^{-12}(\omega l)^2,$$

$$k_2 = \frac{\omega^2 l^2 \rho_0}{4\mu} \sim 2.5 \cdot 10^{-12}(\omega l)^2.$$

Thus, the requirement that k_1 and k_2 be small nevertheless enables us to investigate a wide (unlike the diffraction problems) and most important range of frequencies considered here (at least up to $\omega l \sim 10^5$).

For small values of k_1 and k_2, we can use asymptotic formulas for calculating Mathieu functions, as well as the following well-known relation

$$\mathrm{Fek}'_{2m}(0, k_1) = -\frac{1}{2}\mathrm{Fey}'_{2m}(0, k_1) = -\frac{1}{\pi}\frac{p_{2m}}{A_0^{(2m)}}\mathrm{ce}_{2m}\left(\frac{\pi}{2}, k_1\right),$$

$$\mathrm{Gek}_{2m+2}(0, k_2) = -\frac{1}{2}\mathrm{Gey}_{2m+2}(0, k_2) = \frac{1}{\pi}\frac{s_{2m+2}\mathrm{se}'_{2m+2}\left(\frac{\pi}{2}, k_2\right)}{k_2 B_2^{(2m+2)}},$$

$$\mathrm{Fek}_0(0, k_1) \approx -\frac{\sqrt{2}}{4\pi}\ln k_1,$$

$$\mathrm{Fek}_{2m}(0, k_1) = -\frac{1}{2}\mathrm{Fey}_{2m}(0, k_1) + \frac{i}{2}\mathrm{ce}_{2m}(0, k_1)$$

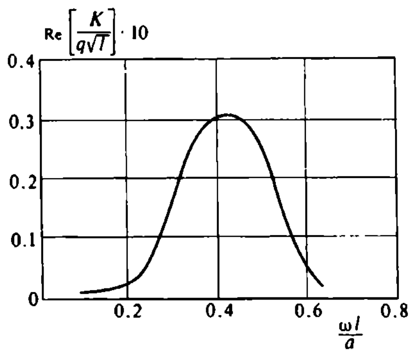

Fig. 32. Dependence of coefficient K on frequency ω.

$$\approx -\frac{1}{2\pi}2^{4m-1}(2m-1)!\,(2m)!\,\frac{1}{k_1^{2m}}+\frac{i}{2}\,\mathrm{ce}_{2m}(0,k_1),$$

$$\mathrm{Gek}'_{2m+2}(0,k_2) = -\frac{1}{2}\mathrm{Gey}'_{2m+2}(0,k_2)+\frac{i}{2}\,\mathrm{se}'_{2m+2}(0,k_2)$$

$$\approx -\frac{1}{2\pi}2^{4m+3}\left[(2m+2)!\right]^2\frac{1}{k_2^{2m}}+\frac{i}{2}\,\mathrm{se}'_{2m+2}(0,k_2).$$

The results of the numerical calculation of the dimensionless quantity $|K/q\sqrt{l}|$ as a function of $\omega l/a$ are shown in Fig. 32 for $q(\theta) = q = \mathrm{const}$.

The result obtained here is confirmed in [16], where the problem of the incidence of a wave on a cut of finite length is considered.

Chapter Five

Application of Analytical Functions in Plane Problems

Section 1
Torsion of Bars

It was shown in Sec. 3, Ch. 3, Vol. 1 that the problem of torsion of bars can be reduced to a determination in the domain occupied by the cross section, either of a harmonic function $\varphi(x, y)$ called the torsional function and assuming on the contour a given value of the normal derivative or of a harmonic function $\psi(x, y)$, assuming a given value on the contour.

Let us now set up the complex torsional function [17]

$$F(z) = \varphi(x,y) + i\psi(x,y). \tag{1.1}$$

Using the relation (3.4), Ch. 3, Vol. 1, we can represent the stresses in the form

$$\tau_{xz} - i\tau_{yz} = \mu\tau[F'(z) - i\bar{z}]. \tag{1.2}$$

In this case, if we take into account Eq. (3.13), Ch. 3, Vol.1, it is possible to write the boundary conditions in the form

$$i[\overline{F(t)} - F(t)] = t\bar{t} + C_k. \tag{1.3}$$

We assume that the section of the bar is a simply-connected domain D^+. Let the function $z = \omega(\zeta)$ perform a conformal mapping of a unit circle in the ζ-plane onto D^+. We carry out a substitution of variables in the expression for $F(z)$ and denote the function thus obtained by $f(\zeta)$. The boundary condition (1.3) can be rewritten in the following form:

$$i[\overline{f(\sigma)} - f(\sigma)] = t\bar{t} = \omega(\sigma)\overline{\omega(\sigma)}. \tag{1.4}$$

We multiply both sides of this equation by $\dfrac{1}{2\pi i}\dfrac{d\sigma}{\sigma-\zeta}$ ($|\zeta| < 1$) and carry out integration over the circle $|\sigma| = 1$ (action of the Cauchy operator). In accordance with Cauchy's theorem, the integral of the first term on the left-hand side restores the function $f(\zeta)$. In order to calculate the integral of the second term, we use the formula (1.75), Ch. 1, Vol. 1. It turns out that the integral vanishes (the constant a_0 can be neglected because it does not affect the state of stress). Finally, we get

$$f(\zeta) = \frac{1}{2\pi} \int\limits_{|\sigma|=1} \frac{\omega(\sigma)\overline{\omega(\sigma)}}{\sigma - \zeta}\, d\sigma, \tag{1.5}$$

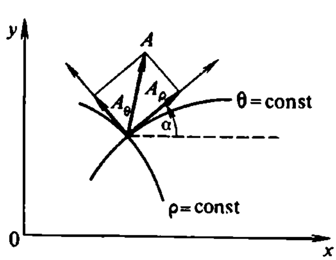

Fig. 33. Curvilinear coordinates corresponding to conformal mapping.

which leads directly to the solution of the problem. If the function $\omega(\zeta)$ is rational, the value of the integral (1.5) may be obtained explicitly.

In order to determine the tangential stresses, we should turn to the formula (1.2), after going over to the variable z. Of greatest interest is the shearing stress in a direction parallel to the contour. We shall obtain the required formula without performing any rotation of coordinate axes. We introduce curvilinear coordinates which, upon conformal mapping, correspond to a set of concentric circles $\rho =$ $=$ const and to a beam of straight lines $\theta =$ const, passing through the origin of the coordinates. Let A be an arbitrary vector with components A_x and A_y in Cartesian coordinates. We denote the same components by A_ρ and A_θ in curvilinear coordinates. The following equality is then obvious:

$$A_\rho + iA_\theta = e^{-i\alpha}(A_x + iA_y), \tag{1.6}$$

where α is the angle between the positive direction of the axis corresponding to the line $\theta =$ const and the x-axis (Fig. 33). In order to determine the quantity $e^{-i\alpha}$, we proceed as follows. We displace the point z by an amount dz along the indicated axis. Then the corresponding point ζ will shift in the radial direction by an amount $d\zeta$. Using the equalities $dz = e^{i\alpha}|dz|$ and $d\zeta = e^{i\theta}|d\zeta|$, we find that

$$e^{-i\alpha} = \frac{|dz|}{dz} = \frac{|\omega'(\zeta)|}{\omega'(\zeta)}\frac{|d\zeta|}{d\zeta} = e^{-i\theta}\frac{|\omega'(\zeta)|}{\omega'(\zeta)} = \frac{\rho}{\zeta}\,\frac{|\omega'(\zeta)|}{\omega'(\zeta)}. \tag{1.7}$$

Considering the case under investigation, we obtain the required formula

$$T_\rho - iT_\theta = \frac{\zeta}{\rho}\,\frac{\omega'(\zeta)}{|\omega'(\zeta)|}(\tau_{xz} - i\tau_{yz}) = \frac{\mu\tau\zeta}{\rho|\omega'(\zeta)|}\left[f'(\zeta) - i\;\omega(\zeta)\;\omega'(\zeta)\right]. \tag{1.8}$$

By way of an example, let us consider the case when the domain D^+ is bounded by an epitrochoid whose parametric equation is given by (n is an integer)

$$\begin{aligned} x &= R(\cos\theta + m\cos n\theta) \\ y &= R(\sin\theta + m\sin n\theta) \quad \left(R > 0, 0 \leqslant m \leqslant \frac{1}{n}\right). \end{aligned} \tag{1.9}$$

The function performing the conformal mapping has the form

$$z = \omega(\zeta) = R(\zeta + m\zeta^n). \tag{1.10}$$

The numerator of the integrand in the Cauchy-type integral (1.5) can be written in the following form:

$$\omega(\sigma)\overline{\omega(\sigma)} = R^2\left(1 + m^2 + m\sigma^{n-1} + \frac{m}{\sigma^{n-1}}\right). \tag{1.11}$$

Taking this into account, we get

$$f(\zeta) = i\frac{R^2}{m}\zeta^{n-1}. \tag{1.12}$$

We shall give the final expression only for the component T_θ on the contour for $\rho = 1$ (the component T_ρ, naturally, vanishes):

$$T_\theta = \mu\tau Rm \frac{n + \dfrac{2}{m}\cos(n-1)\theta + \dfrac{1}{m^2}}{\sqrt{n^2 + 2\dfrac{n}{m}\cos(n-1)\theta + \dfrac{1}{m^2}}}. \tag{1.13}$$

The maximum value of the shearing stress, attained at the point with angular coordinate θ, which is the root of the equation $\cos(n-1)\theta = -1$, is equal to

$$T_{\max} = \mu\tau R\frac{1 - m + nm^2}{1 - nm}. \tag{1.14}$$

If the parameter m tends to $1/n$, the stress at this point (which approaches the cusp) indefinitely increases.

It has been mentioned in Sec. 3, Ch. 3, Vol. 1 that a solution of the problems of bending of bars can also be reduced to a solution of harmonic problems, and hence the application of complex variables for considering such problems does not offer anything more than what is contained in the above discussion.

Section 2
The Plane Problem in the Theory of Elasticity. Bending of Plates

Let us rewrite Hooke's law for plane strain and generalized plane state of stress, having represented the stresses in terms of Airy's function

$$\lambda\theta + 2\mu\frac{\partial u}{\partial x} = \frac{\partial^2 U}{\partial y^2}, \qquad \lambda\theta + 2\mu\frac{\partial v}{\partial y} = \frac{\partial^2 U}{\partial x^2},$$
$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -\frac{\partial^2 U}{\partial x \partial y}. \tag{2.1}$$

Substituting into the first two relations the expressions for θ in terms of Airy's stress function $\theta = \dfrac{1}{2(\lambda + \mu)}\Delta U$, we get

$$2\mu \frac{\partial u}{\partial x} = \frac{\partial^2 U}{\partial y^2} - \frac{\lambda}{2(\lambda + \mu)}P,$$
$$2\mu \frac{\partial v}{\partial y} = \frac{\partial^2 U}{\partial x^2} - \frac{\lambda}{2(\lambda + \mu)}P, \tag{2.2}$$

where the notation $P = \Delta u$ has been introduced. Since

$$\frac{\partial^2 U}{\partial y^2} = P - \frac{\partial^2 U}{\partial x^2}, \qquad \frac{\partial^2 U}{\partial x^2} = P - \frac{\partial^2 U}{\partial y^2},$$

we get, introducing these equations into (2.2),

$$2\mu \frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)}P,$$
$$2\mu \frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)}P. \tag{2.3}$$

Next, we denote by Q the function which is conjugate to the harmonic function P. This permits the introduction of the analytical function

$$f(z) = P(x, y) + iQ(x, y),$$

as well as the analytical function

$$\varphi(z) = \frac{1}{4}\int f(z)\, dz = p + iq.$$

From the last two relations, we get $\varphi'(z) = (1/4)f(z)$. This relation can be written more explicitly as follows:

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{1}{4}P, \qquad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} = -\frac{1}{4}Q.$$

We shall now transform the relations (2.3) with the help of the functions introduced above:

$$2\mu \frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}\frac{\partial p}{\partial x},$$
$$2\mu \frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}\frac{\partial q}{\partial y}.$$

Integrating, we get

$$2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}p + f_1(y),$$

$$2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}q + f_2(x),$$

where $f_1(y)$ and $f_2(x)$ are arbitrary functions. Substituting these expressions into the third relation in (2.1), we get

$$\frac{\lambda + 2\mu}{\lambda + \mu}\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) + [f_1'(y) + f_2'(x)] = 0.$$

Since the functions p and q are conjugate, the first term is equal to zero, and hence the second term is also equal to zero. In view of this, the functions $f_1(y)$ and $f_2(x)$ may be represented as follows:

$$f_1 = -\varepsilon y + \alpha, \qquad f_2 = \varepsilon x + \beta,$$

which corresponds to the displacement of the body as a rigid entity (ε, α,and β are certain constants which can be put equal to zero). Finally, we get

$$\begin{aligned} 2\mu u &= -\frac{\partial U}{\partial x} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}p, \\ 2\mu v &= -\frac{\partial U}{\partial y} + \frac{2(\lambda + 2\mu)}{\lambda + \mu}q. \end{aligned} \tag{2.4}$$

Let us now construct a new function $p_1 = U - px - qy$. It can be shown with the help of direct calculations (using the relation between $\varphi(z)$ and $f(z)$) that this function is harmonic. This gives

$$U(x, y) = px + qy + p_1.$$

We introduce another function $\chi(z)$, whose real part is the function p_1. The last relation can then be represented in terms of two analytical functions of a complex variable:

$$U(x,y) = \mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)]. \tag{2.5}$$

The expression (2.5) is called Goursat's formula.

The following line of reasoning will be connected with the representation of displacements and stresses in terms of these functions $\varphi(z)$ and $\chi(z)$. We differentiate the left- and the right-hand side of (2.5) with respect to the arguments x and y, and form a new function

$$f(x, y) = \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}.$$

Unlike the individual expressions for the derivatives $\partial U/\partial x$ and $\partial U/\partial y$, the expres-

sion for the sum $\partial U/\partial x + i\partial U/\partial y$ has a fairly compact form:

$$f(x,y) = \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2.6}$$

where a new function $\psi(z) = \chi'(z)$ has been introduced in order to simplify the notation. Using the relations (2.4) and (2.6), we get the following expression for the combination $u + iv$:

$$2\mu(u + iv) = -[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}] + \frac{2(\lambda + 2\mu)}{\lambda + \mu}\varphi(z)$$
$$= \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}. \tag{2.7}$$

The constant $\varkappa$ is equal to $3 - 4\nu$ in the case of plane strain, while for plane state of stress, $\varkappa = (3 - \nu)/(1 + \nu)$.

In order to express the stresses in terms of analytical functions of a complex variable, we differentiate the expressions for $\partial U/\partial x$ and $\partial U/\partial y$ once again. The relations obtained in this way are quite cumbersome, but it is possible to represent them in a compact form

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\,[\Phi(z)],$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \tag{2.8}$$
$$(\Phi(z) = \varphi'(z),\ \Psi(z) = \psi'(z)).$$

In this form, we can also represent the stress components in a system of coordinates, inclined at a certain angle α with respect to the original system. Using the formulas (1.19), (1.20), and (1.22), Ch. 2, Vol. 1, it can be shown that the following equalities hold:

$$\sigma_x' + \sigma_y' = 4\,\mathrm{Re}\,[\Phi(z)],$$
$$\sigma_y' - \sigma_x' + 2i\tau_{xy}' = 2[\bar{z}\Phi'(z) + \Psi(z)]e^{2i\alpha}. \tag{2.8'}$$

Combining these expressions, we arrive at a useful formula

$$\sigma_y' + i\tau_{xy}' = \Phi(z) + \overline{\Phi(z)} - e^{2i\alpha}[\bar{z}\Phi'(z) + \Psi(z)]. \tag{2.8''}$$

The relations (2.8), (2.8′), and (2.8″) are called the Kolosov-Muskhelishvili formulas [17, 18]. In future, we shall use the pair of functions $\varphi(z)$, $\psi(z)$, as well as the pair $\Phi(z)$, $\Psi(z)$. Thus, in order to determine the shearing stress, we must find the imaginary part of the second relation in (2.8), while for normal stresses, the corresponding second-order system must be solved.

Let us now consider a small arc ds in the region occupied by an elastic body, and try to determine the stress vector applied to this arc. Let $l = \cos(n, x)$ and $m = \cos(n, y)$ be the direction cosines in the positive direction of the normal to the arc ds. Since $l = dy/ds$ and $m = -dx/ds$, we get the following expressions, in accordance with the formulas (1.6), Ch. 2, Vol. 1, for the projections of the stress

vector:

$$\sigma_{x\nu} = \sigma_x l + \tau_{xy} m = \frac{\partial^2 U}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 U}{\partial x \partial y}\frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial U}{\partial y}\right),$$

$$\sigma_{y\nu} = \tau_{xy} l + \sigma_y m = -\frac{\partial^2 U}{\partial x \partial y}\frac{dy}{ds} - \frac{\partial^2 U}{\partial y^2}\frac{dx}{ds} = -\frac{d}{ds}\left(\frac{\partial U}{\partial x}\right).$$

The combination $\sigma_{x\nu} + i\sigma_{y\nu}$ may then be represented in the form

$$\sigma_{x\nu} + i\sigma_{y\nu} = \frac{d}{ds}\left(\frac{\partial U}{\partial y} - i\frac{\partial U}{\partial x}\right)$$

$$= -i\frac{d}{ds}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right) = -i\frac{d}{ds}[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}].$$

Let us now consider an arc of finite length having its ends at the points A and B. We shall determine the vector of forces $P_x + iP_y$ applied to it from the side of the positive normal:

$$P_x + iP_y = \int_{AB} (\sigma_{x\nu} + i\sigma_{y\nu})\, ds = -i\,[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]_A^B. \tag{2.9}$$

We fix the point A and denote the corresponding point on the arc s through z_0. The point B is assumed to be varying. This leads to the following dependence which we shall find useful at a later stage:

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \int_{z_0}^{z} (\sigma_{x\nu} + i\sigma_{y\nu})\, ds + \text{const.} \tag{2.10}$$

It can be shown that the magnitude of the principal moment of forces applied to the arc AB with respect to the origin of the coordinates is equal to

$$M = \operatorname{Re}\,[\chi(z) - z\psi(z) - z\bar{z}\varphi'(z)]_A^B. \tag{2.11}$$

Let us now analyze the degree of definiteness of the functions introduced above. As a matter of fact, the quantities having a physical sense, i.e. the displacements and stresses, must be single-valued in the domain occupied by the elastic body.

We turn to the Kolosov-Muskhelishvili formulas (2.8) and assume that the domain under consideration is finite and simply connected. It follows from the first equality that the function $\Phi(z)$ is determined to within the term Ci, where C is a real constant, while the second equation indicates that the function $\Psi(z)$ is determined uniquely. Hence the functions $\varphi(z)$ and $\psi(z)$ are determined to within terms of the type $Ciz + \gamma$ and γ' (where γ and γ' are arbitrary complex constants). Having admitted this ambiguity in determining the functions $\varphi(z)$and $\psi(z)$, we naturally find that the expressions for displacements are also ambiguous. In order to ascertain this, we consider the formulas (2.4) and substitute in these $\varphi_0 = Ciz + \gamma$ and $\psi_0 = \gamma'$. This gives the following expression for the corresponding displacements u and v:

$$(\Delta u + i\,\Delta v) = 2\mu\,[(\varkappa + 1)Ciz + \varkappa\gamma - \bar{\gamma}'\,]. \tag{2.12}$$

Obviously, the displacements Δu and Δv correspond to the displacement of the body as a rigid entity. In order that the displacements be unique, we must put the constant $C = 0$, while the constants γ and γ' must be connected by the relation $\varkappa\gamma = \bar{\gamma}'$.

Let us now assume that the contour L is a closed contour and that the functions $\varphi(z)$ and $\psi(z)$ are single-valued to within the terms indicated above. Then, considering the expressions (2.9) and (2.11), we find that, as expected, the principal vector and the vector moment of the forces applied to the domain bounded by the contour L will vanish.

Non-trivial results are obtained when we consider a multiply connected domain, i.e. a domain bounded by a set of contours $L_0, L_1, L_2, \ldots, L_m$, of which $L_1, L_2, \ldots, L_m$ are outside one another, while the contour L_0 envelops all the remaining contours. As before, the components of the stress tensor and of the displacement vector are assumed to be single-valued functions. A physical interpretation has been given in [17] for the case when the displacements are assumed to be non-single-valued functions. Then, from the condition of uniqueness of the expression $\operatorname{Re}\Phi(z)$ it follows that the function $\Phi(z)$ itself can be defined to within a term by the sum

$$\sum_{k=1}^{m} A_k \ln(z - z_k). \tag{2.13}$$

Here, z_k are certain points inside the corresponding contours L_k, and $A_k = B_k/2\pi$, where B_k is the increment in the imaginary part of the function $\Phi(z)$ as we go round the contour L_k in the anticlockwise direction. Thus, the function $\Phi(z)$ can be represented in the following form:

$$\Phi(z) = \sum_{k=1}^{m} A_k \ln(z - z_k) + \Phi^*(z), \tag{2.14}$$

where $\Phi^*(z)$ is now a single-valued function, analytical in the domain under consideration. Direct integration shows that the function $\varphi(z)$ can be represented as follows:

$$\varphi(z) = z \sum_{k=1}^{m} A_k \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k \ln(z - z_k) + \varphi^*(z), \tag{2.14'}$$

where γ_k are arbitrary complex constants, the coefficients A_k have been defined above, and the function $\varphi^*(z)$ is single-valued in the domain under consideration. From the second relation in (2.8), it follows that the function $\Psi(z)$ is single-valued. Integrating, we find that the function $\psi(z)$ has the following form:

$$\psi(z) = \sum_{k=1}^{m} \gamma_k' \ln(z - z_k) + \psi^*(z), \tag{2.15}$$

where γ_k' are arbitrary complex constants, and $\psi^*(z)$ is a single-valued function.

Let us consider the displacements once again. Going arround each contour L_k, we obtain the increment in displacements in the form

$$2\mu(\Delta u + i\Delta v)|_{L_k} = 2\pi i[(\varkappa + 1)A_k z + \varkappa\gamma_k + \bar{\gamma}_k']. \tag{2.16}$$

The following equalities emerge from the uniqueness of displacements:

$$A_k = 0, \varkappa\gamma_k + \bar{\gamma}_k' = 0. \tag{2.17}$$

Before proceeding further, we must determine the principal vector $P_{xk} + iP_{yk}$ of the forces applied to the contour L_k. Substituting the expressions for $\varphi(z)$ and $\psi(z)$ into (2.9) (passing in the clockwise direction) and neglecting terms corresponding to the single-valued functions $\varphi^*(z)$ and $\psi^*(z)$, we get

$$P_{xk} + iP_{yk} = -2\pi(\gamma_k - \bar{\gamma}_k'). \tag{2.18}$$

Assuming that the components of the principal force vector are given, we obtain the required expressions for the arbitrary complex constants γ_k and γ_k':

$$\gamma_k = -\frac{P_{xk} + iP_{yk}}{2\pi(1 + \varkappa)}, \qquad \gamma_k' = \frac{\varkappa(P_{xk} - iP_{yk})}{2\pi(1 + \varkappa)}. \tag{2.19}$$

Substituting these values into the formulas (2.14) and (2.15), we obtain the following expressions for the functions $\varphi(z)$ and $\psi(z)$:

$$\begin{aligned} \varphi(z) &= -\frac{1}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (P_{xk} + iP_{yk}) \ln(z - z_k) + \varphi^*(z), \\ \psi(z) &= \frac{\varkappa}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (P_{xk} - iP_{yk}) \ln(z - z_k) + \psi^*(z). \end{aligned} \tag{2.20}$$

Finally, let us consider the case when the contour L_0 is missing, i.e. the domain occupied by the elastic body is infinite (it has been indicated above that such a domain may be obtained, for example, by withdrawing the contour L_0 to infinity). Then, for any point z lying outside a certain circle having its centre at the origin of the coordinates and encompassing all the contours L_m, the following equality is valid:

$$\ln(z - z_k) = \ln z + \ln\left(1 - \frac{z_k}{z}\right).$$

It should be noted that the last term outside this circle is a single-valued function. Hence we can transform the functions $\varphi(z)$ and $\psi(z)$ as follows:

$$\varphi(z) = \frac{P_x + iP_y}{2\pi(1+\varkappa)} \ln z + \varphi^{**}(z),$$

$$\psi(z) = \frac{\varkappa(P_x - iP_y)}{2\pi(1+\varkappa)} \ln z + \psi^{**}(z),$$

where the functions $\varphi^{**}(z)$ and $\psi^{**}(z)$ are single-valued outside the circle, with the possible exception of the point at infinity, and the constants P_x and P_y are the resultant components of the principal vectors of external forces applied to all the contours L_k. Naturally, the behaviour of the functions $\varphi^{**}(z)$ and $\psi^{**}(z)$ at an infinitely remote point must correspond to the admissible stresses at infinity. We shall assume that the stresses at infinity tend to constant values. Then in the expressions for the functions $\varphi^{**}(z)$ and $\psi^{**}(z)$, we can isolate terms of the type Γz and $\Gamma' z$:

$$\Gamma = B + iC, \Gamma' = B' + iC', B = \frac{1}{4}(\sigma_x^\infty + \sigma_y^\infty),$$

$$\sigma_y^\infty - \sigma_x^\infty + 2i\tau_{xy}^\infty = 2(B' + iC'),$$

where B, C, B', and C' are constants.

The connection between these constants and the stresses at infinity may be represented in the following form:

$$\sigma_x^\infty = 2B - B', \quad \sigma_y^\infty = 2B + B', \quad \tau_{xy}^\infty = C'. \tag{2.21}$$

The functions φ^* and ψ^* will tend to constant values at infinity (their magnitude, naturally, does not affect the values of the resulting stresses).

Let us now formulate the boundary value problems directly for the functions $\varphi(z)$ and $\psi(z)$. We start with the first fundamental problem. The condition of the continuity of displacements right up to the boundary is equivalent to the condition of continuous extendability of the expression (2.7) to all points on the boundary. Going over to the boundary points on the left- and right-hand sides of Eq. (2.7), we get

$$-\varkappa\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t). \tag{2.22}$$

The function $f(t)$ must be known.

In the case of the second fundamental problem, we can start from Eq. (2.10), assuming that the points z tend to the boundary points, while the arc along which the integration is carried out is a part of the boundary. This gives

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = \int_{s_0}^{s} (\sigma_{x\nu} + i\sigma_{y\nu})\, ds + c = f(t) + c. \tag{2.23}$$

It follows from Eq. (2.23) that if the principal vector of forces applied to the contour is equal to zero, the function $f(t)$ is single-valued. It can be shown that the equality of the principal vector to zero is expressed by the condition

$$\mathrm{Re} \int_L \overline{f(t)}dt = 0. \tag{2.23'}$$

If the domain is multiply connected, the constants c are not arbitrary (only one of these may be chosen arbitrarily), and they should be determined in the course of the solution from the condition of uniqueness of the displacements obtained.

For the case of a multiply connected domain or the appearance of one contour $f(t)$ may turn out to be multiple-valued in view of the fact that $\varphi(z)$ and $\psi(z)$ may

contain multiple-valued functions (see (2.20)). Hence we must go over to single-valued functions $\varphi^*(z)$ and $\psi^*(z)$, and then the boundary conditions will accordingly be unique.

Of course, if the domains contain infinite points, the formulation of the problems must include conditions for the behaviour of a solution at infinity, given by the formulas (2.21).

It is clear that a combination of the conditions (2.23) and (2.22) can be used to describe a mixed problem.

Let us consider an important question. Generally speaking, only a continuous extendability to the boundary of the expressions on the left-hand sides of (2.22) and (2.23) was required while formulating the boundary value problems. In future, we shall require that a more stringent condition be satisfied, i.e. each term in $\varphi(z)$, $\varphi'(z)$, and $\psi(z)$ be continuously extendable to the boundary. The solution satisfying these conditions will be called regular. The additional condition introduced above considerably simplifies the justification of the methods usually employed for solving boundary value problems by using a complex variable.

It should be noted that the boundary conditions for the second fundamental problem may be directly formulated in terms of stresses. For this purpose, we proceed from the representation (2.8″), assuming that α is the angle between the normal to the contour and the x-axis:

$$\Phi(t) + \overline{\Phi(t)} - e^{2i\alpha}[\bar{z}\Phi'(t) + \Psi(t)] = \sigma_{x\nu} + i\sigma_{y\nu}. \tag{2.24}$$

The uniqueness theorems, established earlier in Ch. 3, Vol. 1 for the general case of a three-dimensional problem, continue to be valid for the plane problem in the theory of elasticity.

Let us now consider the question of an explicit determination of the function $f(t)$, starting from the expression (2.10). We assume that a constant load p, directed along the normal, is given at the segment of an arc. In this case,

$$\sigma_{x\nu} + i\sigma_{y\nu} = -p(l + im) = pi\left(\frac{dx}{ds} + i\frac{dy}{ds}\right) = pi\frac{dt}{ds}$$

and hence

$$f(t) = i\int_{t_0}^{t}(\sigma_{x\nu} + i\sigma_{y\nu})dt = -pt + c. \tag{2.25}$$

We now assume that a certain contour has three sections. In the first section (from point t_0 to t_1), there is no load, in the second section (up to the point t_2) a hydrostatic pressure p is given, while in the third section (up to the point t_3) the load is again absent. Then, we find that $f(t) = 0$ in the first section (the constant has been put equal to zero), in the second section, $f(t) = -pt + pt_1$ (the corresponding constant has been determined from the condition of continuity of the function $f(t)$ at the point t_1), while in the third section, $f(t) = p(t_1 - t_2)$. Let us now proceed to the limit, drawing the points t_2 and t_1 towards each other and simultaneously increasing p, so that the product $p|t_2 - t_1|$ remains constant. The values of the function $f(t)$ in the first and third sections will not change, while in the limit, we get a discon-

tinuous function. Obviously, the value of the discontinuity may be treated as a concentrated force applied along the normal to the contour. In the general case, if the components of the jump are arbitrary, we get an expression for an arbitrarily directed force. The condition that the load be constant on the section t_2, t_1 was introduced, of course, in order to simplify the calculations.

It was mentioned in Sec. 4, Ch. 3, Vol. 1 that the problem of bending of plates can be reduced, generally speaking, to the solution of an inhomogeneous biharmonic equation. If a particular solution of the inhomogeneous equation has been obtained somehow, we arrive at the solution of a homogeneous equation now. In order to solve the corresponding boundary value problems, it is naturally convenient to use complex variables. We shall now write down the necessary formulas (their derivation is practically identical to the corresponding constructions in the case of a plane problem). For the force parameters M_x, M_y, M_z, Q_x, and Q_y, the following formulas hold:

$$\begin{aligned} M_y - M_x + 2iM_{xy} & \\ &= 4D(1-\nu)[\bar{z}\varphi''(z) + \psi'(z)] + (M_y^1 - M_x^1 - 2iM_{xy}^1), \qquad (2.26)\\ M_x + M_y &= -4D(1+\nu)[\varphi'(z) + \overline{\varphi'(z)}] + (M_x^1 + M_y^1),\\ Q_x - iQ_y &= -8D\varphi''(z) + (Q_x^1 - iQ_y^1). \end{aligned}$$

Here and onwards, the superscript "1" on the functions indicates that the corresponding quantities are defined by the particular solution.

In the case when the edge of a plate is free from geometrical constraints and hence the bending moment $f_1(t)$ and the shearing force $f_2(t)$ are given, we get the boundary condition

$$\begin{aligned} -\, x\overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z) & \\ &= \frac{1}{2D(1-\nu)}\left[\int_0^s (f_1^1 + if_2^1)ds - \int_0^s (f_1 + if_2)ds\right] - icz, \qquad (2.27) \end{aligned}$$

where c is an arbitrary constant.

If, however, the magnitude of deflection $w = f_1$ and the angle of rotation $\partial w/\partial n = f_2$ are given, we get the condition

$$\begin{aligned} \overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z) &= \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial w^1}{\partial x} - i\frac{\partial w^1}{\partial y}\right),\\ \frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y} &= \left(f_2 + i\frac{df_1}{ds}\right)e^{-i\alpha}. \end{aligned} \qquad (2.28)$$

Other combinations of boundary conditions are more cumbersome to write. For example, when deflections and the bending moment are given, we obtain

$$\begin{aligned} \mathrm{Re}\left\{x_1^*\varphi'(t) - \left(\frac{dt}{ds}\right)^2[\bar{t}\varphi''(t) + \varphi'(t)]\right\} &= f_1(t),\\ \mathrm{Re}\left\{\frac{dt}{ds}\left[\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t)\right]\right\} &= f_2(t) \quad \left(x^* = \frac{2(1+\nu)}{1-\nu}\right). \end{aligned} \qquad (2.29)$$

It is interesting to note that if we change to the conjugate values in the conditions (2.26) and (2.27), we get the conditions (2.22) and (2.23) for the plane problem.

Let us consider the following plane problem [19]. Suppose that a certain multiply connected domain D_0 is bounded by smooth contours L_j ($j = 0, 1, 2, \ldots, m$), of which all the contours L_j ($j \neq 0$) are situated outside one another, while the contour L_0 (which may not be present) envelops all the other contours. The domains D_0 and D_j^+ ($j = 1, 2, \ldots, m$) are filled with an elastic medium having the same values of Lamé's constants as the domain D (i.e. λ and μ). For each of these domains, we introduce the functions $\varphi_j(z)$ and $\psi_j(z)$.

The following conjugation conditions are satisfied on the contours L_j:

$$\varphi_j(t) + t\overline{\varphi_j'(t)} + \overline{\psi_j(t)} = \varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} \quad (t \in L_j,\ j \neq 0), \tag{2.30}$$

$$\varkappa\varphi_j(t) - t\overline{\varphi_j'(t)} - \overline{\psi_j(t)} = \varkappa\varphi_0(t) - t\overline{\varphi_0'(t)} - \overline{\psi_0(t)} + 2\mu\delta_j(t), \tag{2.31}$$

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = f(t)\ (t \in L_0), \tag{2.32}$$

which express the equation for the stress vector and the presence of a given jump $\delta_j(t)$ in the displacement vector. These conditions are treated as a problem about tight fitting [19], which is of great practical importance, since by a suitable choice of the displacement jump, we can, as a rule, increase the bearing capacity of a construction.

We assume that the condition (2.10) is valid on the contour.

Conditions (2.30) and (2.31) lead to the following equations:

$$\varphi_j(t) = \varphi_0(t) + \frac{1}{1+\varkappa}\delta_j^1(t) \quad (t \in L_j, j \neq 0), \tag{2.33}$$

$$\psi_j(t) = \psi_0(t) + \frac{1}{1+\varkappa}[\overline{\delta_j^1(t)} + \bar{t}\delta_j^{1\prime}(t)]\ (\delta_1^j = 2\mu\delta_j(t)). \tag{2.34}$$

For each of the domains D_j^+ ($j \neq 0$), we now form the functions

$$\varphi_j^*(z) = \varphi_j(z) - \frac{1}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\delta_k^1(t)}{t-z}dt. \tag{2.35}$$

Although all the components in this equation are identical in appearance, yet it should be borne in mind that the points z belong to different domains. Let us find the expressions for the limiting values of these functions on the contours L_j ($j \neq 0$). The limiting values for the functions φ_j^*($j \neq 0$) are determined from inside, while those for the functions φ_0^*, from outside. In accordance with (1.14), Ch. 1, Vol. 1, we have

$$\varphi_j^{*+}(t_0) = \varphi_j^+(t_0) + \frac{1}{1+\varkappa}\left[-\frac{1}{2}\delta_j^1(t_0) - \sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\delta_k^1(t)}{t-t_0}dt\right], \tag{2.36}$$

$$\varphi_0^{*-}(t_0) = \varphi_0^-(t_0) + \frac{1}{1+\varkappa}\left[\frac{1}{2}\delta_j^1(t_0) - \sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\delta_k^1(t)}{t-t_0}dt\right]. \tag{2.37}$$

Then, from (2.33) (strictly speaking, the values of $\varphi_j^+(t)$ and $\varphi_j^-(t)$ are implied in these equations), we find that the functions $\varphi_j^*(z)$ and $\varphi_0^*(z)$ coincide on the contours $L_j (j \neq 0)$. Since each of the functions $\varphi_j^*(z)$ is analytical in the respective domains, it follows that all of them represent a single function which is analytical in the total domain $D(D_0 \cup D_1^+ \cup D_2^+ \cup \ldots \cup D_m^+)$. Hence we shall omit the subscript in the following description.

In each of the domains D_j^+ $(j \neq 0)$ (as well as in D_0), we now form new functions

$$\psi_j^*(z) = \psi_j(z) + \frac{1}{1+\varkappa}\sum_{k=1}^{m}\frac{1}{2\pi i}\int_{L_k}\frac{\overline{\delta_k^1(t) + \bar{t}\delta_k^1(t)}}{t-z}dt. \tag{2.38}$$

As in the case of the functions $\varphi_j^*(z)$, we find that the functions $\psi_j^*(z)$ also represent a single function which is analytical in the entire domain D. Since all the components in (2.35) and (2.38) are known and can be calculated, the initial problem is now reduced to the problem in the theory of elasticity for a continuous domain D. In order to obtain the corresponding boundary conditions, we consider the condition (2.32), having expressed the functions $\varphi_0(z)$ and $\psi_0(z)$ in terms of $\varphi^*(z)$ and $\psi^*(z)$ in accordance with (2.35) and (2.38). As a result, we arrive at the following boundary value problem:

$$\varphi^*(t) + t\overline{\varphi^{*\prime}(t)} + \overline{\psi^*(t)} = f(t) + H(\delta^1, t). \tag{2.39}$$

Here, $H(\delta^1, t)$ stands for the terms which are determined by all the additional components. By solving this problem, we can reconstruct the functions $\varphi_j(z)$ and $\psi_j(z)$.

Section 3
Regular Integral Equations

We shall simultaneously consider the first and second problems for a finite simply connected domain D^+, bounded by a smooth contour L. The boundary conditions (2.22) and (2.23) can be represented in the form

$$k\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t) = \overline{f(t)}, \tag{3.1}$$

where $k = -\varkappa$ for the problem I^+, and $k = 1$ for the problem II^+.

The condition (3.1) can be expressed in a different form:

$$\psi(t) = \overline{f(t)} - k\overline{\varphi(t)} - \bar{t}\varphi'(t). \tag{3.2}$$

Since the function $\psi(t)$ is analytical in the domain D^+, the right-hand side of Eq. (3.2) is the boundary value of a function which is analytical in the domain D^+. In accordance with Cauchy's integral theorem (1.6), Ch. 1, Vol. 1, we get

$$\frac{1}{2\pi i}\int_L \frac{\overline{f(t)} - k\overline{\varphi(t)} - \bar{t}\varphi'(t)}{t - z}\,dt = 0 \quad (z \notin D^+)$$

or, otherwise,

$$\frac{1}{2\pi i}\int_L \frac{k\overline{\varphi(t)} + \bar{t}\varphi'(t)}{t - z}\,dt = \frac{1}{2\pi i}\int_L \frac{\overline{f(t)}dt}{t - z} = A(z). \tag{3.3}$$

Thus, we have obtained a functional equation for the function $\varphi(t)$. Let us transform it to an integral equation. For this purpose, we perform a limit transition to the points of the contour L, always remaining outside the domain D^+. It is assumed that the functions $\varphi(t)$, $\varphi'(t)$, and $f(t)$ satisfy the H-L conditions. Hence we can use the Sokhotskii-Plemelj formulas (1.14), Ch. 1, Vol. 1, for Cauchy-type integrals, whose densities are given by these functions.

Next, we carry out integration by parts, and in order to obtain the equations in a compact form, we use the following identities:

$$t_0\left(-\frac{1}{2}\overline{\varphi(t_0)} + \overline{\frac{1}{2\pi i}\int_L \frac{\varphi(t)dt}{t - t_0}}\right) = 0,$$

$$k\left(-\frac{1}{2}\varphi'(t_0) + \frac{1}{2\pi i}\int_L \frac{\varphi'(t)dt}{t - t_0}\right) = 0.$$

As a result, we obtain an integral equation called the Muskhelishvili equation:

$$-k\varphi(t_0) - \frac{1}{2\pi i}\int_L \overline{\varphi(t)}d\ln\frac{\bar{t} - \bar{t}_0}{t - t_0} - \frac{1}{2\pi i}\int_L \varphi(t)d\frac{\bar{t} - \bar{t}_0}{t - t_0} = A(t_0). \tag{3.4}$$

This equation belongs to the class of Fredholm's equations. The analysis of this equation consists in establishing the conditions for its solvability, as well as in proving the fact that any of its solutions (the function $\varphi(t)$) must be the boundary value of a function which is analytical in the domain D^+.

Let us consider the second fundamental problem ($k = 1$). Let $\varphi(t)$ be any solution of Eq. (3.4). We form the Cauchy-type integrals

$$i\Phi(z') = \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t - z'}\,dt,$$
$$-i\Psi(z') = \frac{1}{2\pi i}\int_L \frac{\overline{\varphi(t)} + \bar{t}\varphi'(t) - \overline{f(t)}}{t - z'}\,dt \qquad (z' \notin D^+). \qquad (3.5)$$

Then Eq. (3.4) can be treated as a relation between the functions[1] $\Phi(t)$ and $\Psi(t)$, namely

$$\overline{\Phi(t)} + \bar{t}\Phi'(t) + \Psi(t) = 0. \qquad (3.5')$$

The relation (3.5′) is a homogeneous boundary condition for the second fundamental problem in the domain D^-. From the uniqueness theorem it follows that the function $\Phi(z') = i\alpha z' + \beta$ and $\Psi(z') = -\bar{\beta}$, where α is a real constant and β is a complex constant. Since these functions can be represented in terms of Cauchy-type integrals, they must vanish at infinity. Consequently, the functions $\Phi(z')$ and $\Psi(z')$ are identically equal to zero. From the condition of vanishing of $\Phi(z')$ for $z' \in D^-$, it follows that the function $\varphi(t)$ is the boundary value of a function which is analytical in D^+, Q.E.D.

The homogeneous equation (3.4) has a non-trivial solution $\varphi_0(z) = i\alpha z + \beta$ (α and β are a real and a complex constant respectively, as before), since it corresponds to the state of zero stress. It follows from the uniqueness theorem for the solution of the boundary value problem that there are no other eigenfunctions. It should be recalled that the second boundary value problem in the theory of elasticity can be solved when the principal vector and the vector moment of the external forces are equal to zero. The first condition means that the function $f(t)$ is single-valued, while the second condition leads to the equality

$$\operatorname{Re}\int_L \overline{f(t)}dt = 0. \qquad (2.23')$$

A direct realization of the generally accepted scheme, which can be used to establish the solvability conditions for Fredholm's equation (construction of the eigenfunction of the companion equation and checking the orthogonality condition for its right-hand side), is difficult in the present case. A different method for analyzing Eq. (3.4) has been proposed in [20]. The following term is introduced into the left-hand side of this equation:

$$\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}dt + \frac{1}{2\pi i}\frac{1}{t_0}\int_L \left[\frac{\varphi(t)}{t^2}dt + \frac{\overline{\varphi(t)}}{\bar{t}^2}d\bar{t}\right]. \qquad (3.6)$$

[1] These functions are the boundary values of functions which are analytical in D^-.

This leads, generally speaking, to a new integral equation

$$- k\overline{\varphi(t_0)} - \frac{1}{2\pi i}\int_L \overline{\varphi(t)}\,d\ln\frac{\bar t - \bar t_0}{t - t_0} - \frac{1}{2\pi i}\int_L \varphi(t)d\frac{\bar t - \bar t_0}{t - t_0}$$
$$+ \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}dt + \frac{1}{2\pi i}\cdot\frac{1}{t_0}\int_L\left[\frac{\varphi(t)}{t^2}dt + \frac{\overline{\varphi(t)}}{\bar t^2}d\bar t\right] = A'(t_0). \tag{3.4'}$$

If, in analogy with Eq. (3.5), we introduce the function $\Phi(z')$ and also the function $\Psi(z')$ in the form

$$\Psi(z') = \frac{1}{2\pi i}\int_L \frac{\overline{\varphi(t)} + \bar t\varphi'(t) - \overline{f(t)}}{t - z'}dt$$
$$+ \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}dt + \frac{1}{2\pi i}\frac{1}{z'}\int_L\left[\frac{\varphi(t)}{t^2}dt + \frac{\overline{\varphi(t)}}{\bar t^2}d\bar t\right], \tag{3.7}$$

we again arrive at Eq. (3.5) for their limiting values (from outside). In particular, it follows from this that the function $\Psi(z') = 0$ for $z' \in D^-$, and hence each coefficient of its expansion into power series in $1/z'$ must vanish. The first coefficient has the form

$$\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}dt, \tag{3.8}$$

while the second coefficient (of $1/z'$) is given by

$$\frac{1}{2\pi i}\int_L [\overline{\varphi(t)} + \bar t\varphi'(t)]dt - \frac{1}{2\pi i}\int_L \overline{f(t)}dt$$
$$+ \frac{1}{2\pi i}\int_L\left[\frac{\varphi(t)}{t^2}dt + \frac{\overline{\varphi(t)}}{\bar t^2}d\bar t\right]. \tag{3.9}$$

Since

$$\int_L \bar t\varphi'(t)dt = -\int_L \varphi(t)d\bar t,$$

we find that the first term in (3.9) is a real quantity. If we now require that the condition (2.23') be satisfied, the second term is also found to be real. The third term, however, is imaginary. From the condition that the entire sum (3.9) be equal to zero, it follows that this term vanishes.

In this way, we find that any solution of Eq. (3.4′) nullifies all the terms in (3.6) if the condition (2.23′) is satisfied, and is therefore a solution of the original equation (3.4).

The last stage of discussion consists in proving that Eq. (3.4) is solvable for any right-hand side. Suppose that a homogeneous equation has a non-trivial solution which we denote by φ_1. Since the right-hand side vanishes, the condition (2.23′) is automatically satisfied. Hence the function φ_1 must also be a solution of the homogeneous equation (3.4), and consequently must coincide with $\varphi_0(t)$. Substituting this function into (3.6), we find from the condition that each term tends to zero that the constants α and β vanish.

Thus, we have proved that the homogeneous equation (3.4′) does not have any non-trivial solutions, and hence an inhomogeneous equation is solvable for any right-hand side, and in particular, when the condition (2.23′) is satisfied. In the latter case, a solution of Eq. (3.4′) will also be a solution of Eq. (3.4), which proves that the latter equation is solvable.

In the same way, we carry out an analysis in the case of the first fundamental problem ($k = -x$). For this purpose, we must introduce the following functional into the left-hand side of the equation:

$$\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t}\,dt. \tag{3.10}$$

Obviously, a numerical realization of the Muskhelishvili equation is difficult on account of the fact that the equation is arranged in a spectrum. In particular, solution by the method of mechanical quadratures leads to degenerate systems. It has been mentioned in [21] that in any form of realization, the terms (3.6) or (3.10) should be retained. On account of error in the quadrature formulas, these terms will not vanish, generally speaking, and hence they will introduce some (small) error. However, the computational difficulties mentioned above are completely eliminated by using such an approach.

Let us now go over to the construction of integral equations called the Sherman-Lauricella equations [22]. In this case, we can directly consider a multiply connected domain D, bounded by several contours $L_0, L_1, L_2, \ldots, L_m$, of which $L_1, L_2, \ldots, L_m$ are situated outside one another, while the contour L_0 (which may not exist) envelops all the remaining contours. Finite domains bounded by contours L_j ($j = 1, 2, \ldots, m$) will be denoted by D_j^+, while the exterior of the contour L_0 will be represented through D_0^-.

We start with a consideration of the second fundamental problem. It is assumed that the principal vector of forces applied to each of the contours L_j vanishes[2]. The boundary conditions are written in the form

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f_j(t) + c_j \qquad (t \in L_j, j = 0, 1, 2, \ldots, m), \tag{3.11}$$

where f_j are given functions, and c_j are constants to be determined during the course

[2] If this is not so, a certain transformation of the boundary conditions must be carried out.

of solution of the problem. One of these constants can be arbitrarily chosen, hence we put $c_0 = 0$.

We seek the unknown functions $\varphi(z)$ and $\psi(z)$ in the form of the integrals ($L = L_0 \cup L_1 \cup L_2 \cup \ldots \cup L_m$)

$$\varphi(z) = \frac{1}{2\pi i} \int_L \frac{\omega(t)}{t - z} dt, \tag{3.12}$$

$$\psi(z) = \frac{1}{2\pi i} \int_L \frac{\overline{\omega(t)}}{t - z} dt + \frac{1}{2\pi i} \int_L \frac{\omega(t)}{t - z} dt$$

$$- \frac{1}{2\pi i} \int_L \frac{\bar{t}\omega(t)}{(t - z)^2} dt + \sum_{j=1}^{m} \frac{b_j}{z - z_j}. \tag{3.13}$$

Here, $\omega(t)$ is a function to be determined, z_k are arbitrarily chosen points in the domains D_j^+ ($j = 1, 2, \ldots, m$), and b_j are real constants defined as follows:

$$b_j = i \int_{L_j} [\omega(t) d\bar{t} - \overline{\omega(t)} dt]. \tag{3.14}$$

We shall assume that the function $\omega(t)$ satisfies the H-L condition. We now perform the limit transition to the boundary points for the functions $\varphi(z)$ and $\psi(z)$. The transition is carried out from outside for the contours L_j ($j = 1, 2, \ldots, m$), and from inside for the contour L_0. A similar procedure is carried out for the expression for the function $\varphi'(z)$ obtained from (3.12) by successive differentiation and the application of the formula for integration by parts. Substitution of the expressions obtained in this way into the boundary conditions (3.11) leads to the integral Fredholm equation

$$\omega(t_0) + \frac{1}{2\pi i} \int_L \omega(t) d \ln \frac{t - t_0}{\bar{t} - \bar{t}_0} - \frac{1}{2\pi i} \int_L \overline{\omega(t)} d \frac{t - t_0}{\bar{t} - \bar{t}_0}$$

$$+ \sum_{j=0}^{m} \frac{\bar{b}_j}{\bar{t}_0 - \bar{z}_j} = f_j(t) + c_j \quad (t \in L_j). \tag{3.15}$$

This equation is called the Sherman-Lauricella equation. The left-hand side of this equation contains an additional term $\bar{b}_0/(\bar{t}_0 - \bar{z}_0)$, where the point z_0 ($z_0 = 0$) has been retained for uniformity of notation, and

$$b_0 = \int_L \left[\frac{\omega(t)}{t^2} dt + \frac{\overline{\omega(t)}}{\bar{t}^2} d\bar{t}\right]. \tag{3.16}$$

Further, we put

$$b_j = -\int_{L_j} \omega(t)ds. \tag{3.17}$$

The relation (3.15) then turns into an integral equation whose terms are completely defined.

We shall show that any solution of this equation turns the coefficient b_0 to zero if the principal moment of forces applied to the body is equal to zero. This condition may not be satisfied if the contour L_0 is missing, but in that case, the corresponding term also does not exist.

Equation (3.15) is equivalent to the following boundary value problem:

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + \frac{b_0}{\bar{t}} - c_j = f(t) \quad (t \in L_j). \tag{3.18}$$

Multiplying both sides by $d\bar{t}$ and integrating, we get

$$\int_L [\varphi(t)d\bar{t} - \overline{\varphi(t)}dt] - 2\pi i\bar{b}_0 = \int_L f(t)d\bar{t}.$$

All the terms in this equation, except $2\pi i\bar{b}_0$ are imaginary, hence $b_0 = 0$.

We shall prove that if we take Eq. (3.16) into account, Eq. (3.15) is always solvable. We assume that the converse is true. This gives a non-trivial solution for the homogeneous equation, which we denote by $\omega_0(t)$. Let $\varphi_0(z)$, $\psi_0(z)$, and c_j^0 be the functions and constants defined by this solution. Then,

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} - c_j^0 = 0 \quad (t \in L_j). \tag{3.19}$$

It follows from the uniqueness theorem that

$$\varphi_0(z) = i\alpha z + \beta, \qquad \psi_0(z) = -\bar{\beta}, \qquad c_j^0 = 0. \tag{3.20}$$

Taking (3.12) and (3.13) into account, we get

$$i\alpha z + \beta = \frac{1}{2\pi i}\int_L \frac{\omega_0(t)}{t-z}dt,$$

$$-\bar{\beta} = \frac{1}{2\pi i}\int_L \frac{\overline{\omega_0(t)}dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega'_0(t)dt}{t-z} + \sum_{j=1}^{m} \frac{b_j}{z-z_j}. \tag{3.21}$$

Since the terms outside the integrals in these equations are analytical functions in D, these terms may be represented by Cauchy's integrals of their limiting values. Hence, by introducing the functions

$$i\varphi^*(t) = \omega_0(t) - i\alpha t - \beta, \tag{3.22}$$

$$i\psi^*(t) = \overline{\omega_0(t)} - \bar{t}\omega'_0(t) + \sum_{j=1}^{m} \frac{b_j}{t - z_j} \tag{3.23}$$

on the contours L_j, Eqs. (3.21) may be transformed as follows:

$$\frac{1}{2\pi i}\int_L \frac{\varphi^*(t)}{t-z}dt = 0, \frac{1}{2\pi i}\int_L \frac{\psi^*(t)}{t-z}dt = 0 \quad (z \in D). \tag{3.24}$$

It follows from this that the functions $\varphi^*(t)$ and $\psi^*(t)$ are the boundary values of the functions analytical in the domains D_j^+ and D_0^-, and that $\varphi^*(\infty) = 0$ and $\psi^*(\infty) = 0$.

It was shown earlier that if the equilibrium conditions are satisfied, the coefficient $b_0 = 0$. In the present case, this condition is obviously satisfied, and hence $b_0^0 = 0$. Substituting the representation for $\omega_0(t)$ from (3.22) into the expression (3.16) for b_0^0, we get $\alpha = 0$. Further, eliminating $\omega_0(t)$ from (3.22) and (3.23), we get

$$\varphi^*(t) + \bar{t}\varphi^{*\prime}(t) + \psi^*(t) = i\sum_{j=1}^{m} \frac{b_j^0}{\bar{t} - \bar{z}_j} - 2i\bar{\beta}. \tag{3.25}$$

Multiplying both sides of this equality by dt and integrating over each of the contours L_j, we get the equalities

$$\int_{L_j} [\overline{\varphi^*(t)}dt - \varphi^*(t)d\bar{t}] = -2\pi b_j^0, \tag{3.26}$$

from which it follows that $b_j^0 = 0$. From the results obtained in (3.25), it follows that the functions $\varphi^*(z)$ and $\psi^*(z)$ solve the second fundamental problem for each of the domains D_j^+ $(j = 0, 1, \dots, m)$ if the stresses at the boundary are equal to zero. On the other hand, it follows from the conditions at infinity $(\varphi^*(\infty) = \psi^*(\infty) = 0)$ that $\varphi^*(z) = \psi^*(z) = 0$, and hence, $\beta = 0$. Thus,

$$\varphi^*(z) = i\alpha_j(z) + \beta_j, \qquad \psi^*(z) = -\beta_j \qquad (z \in D_j^+),$$

from which we get, on the basis of the formulas (3.22) and (3.23),

$$\omega_0(t) = -\alpha_j t + i\beta_j \quad (t \in L_j, j = 1, 2, \dots, m),$$
$$\omega_0(t) = 0 \qquad (t \in L_0).$$

At the same time, it follows from the condition $b_j^0 = c_k^0 = 0$ (see (3.14) and (3.17)) that $\alpha_j = \beta_j = 0$ as well. Hence $\omega_0(t) \equiv 0$. This proves the solvability of Eq. (3.15) (if we take into account Eq. (3.17)) when the principal vector moment of the external forces is equal to zero.

Let us briefly consider two more integral equations. In [23], the boundary condition is chosen directly in terms of stresses. The condition (2.8″) can be represented in a somewhat modified form:

$$\Psi(t) = [\bar{f}(t) - \Phi(t) - \overline{\Phi(t)}]\frac{d\bar{t}}{dt} - \bar{t}\Phi'(t). \tag{3.27}$$

Further, if we carry out a number of procedures similar to those employed for constructing the Muskhelishvili equation, we obtain an integral equation in the form

$$\overline{\Phi(t_0)} + \frac{1}{2\pi i}\int_L \left(\frac{1}{\bar{t} - \bar{t}_0} - \frac{dt_0}{d\bar{t}_0}\frac{1}{t - t_0}\right)\overline{\Phi(t)}d\bar{t}$$

$$+ \frac{1}{2\pi i}\int_L \left(\frac{1}{t - t_0} - \frac{dt_0}{d\bar{t}_0}\frac{\bar{t} - \bar{t}_0}{(t - t_0)^2}\right)\Phi(t)dt$$

$$= \frac{1}{2}f(t) - \frac{1}{2\pi i}\frac{\overline{dt_0}}{d\bar{t}_0}\int_L \frac{f(t)d\bar{t}}{t - t_0}. \tag{3.28}$$

In analogy with the derivation of the Sherman-Lauricella equation, we can directly introduce the representations for the functions $\varphi(z)$ and $\psi(z)$ in terms of the auxiliary function $\omega(t)$:

$$\begin{aligned} \varphi(z) &= -\frac{1}{2\pi i}\int_L \omega(t)\ln(t - z)ds, \\ \psi(z) &= -\frac{1}{2\pi i}\int_L \overline{\omega(t)}\ln(t - z)\,ds - \frac{1}{2\pi i}\int_L \frac{\bar{t}}{t - z}\omega(t)\,ds. \end{aligned} \tag{3.29}$$

Here, only the case of a simply connected domain has been considered for the sake of simplicity. As a result of the limit transition and several transformations, we obtain an integral equation of the type [24]

$$\omega(t_0) + \frac{1}{2\pi i}\int_L \omega(t)\frac{\partial}{\partial s_0}\left[\ln\frac{\bar{t} - \bar{t}_0}{t - t_0}\right]ds$$

$$+ \frac{1}{2\pi i}\int_L \overline{\omega(t)}\frac{\partial}{\partial s_0}\left[\frac{t - t_0}{\bar{t} - \bar{t}_0}\right]ds = i(\sigma_{x\nu} + i\sigma_{y\nu}). \tag{3.30}$$

Let us consider the following question. We have constructed above regular integral equations for the case of multiply connected domains. Each of the contours in these equations appears in an equivalent way. We shall now discuss the method in which each contour is considered in a different way [25]. Basically, we are talking about doubly connected domains. For the sake of definiteness, we assume that a domain D is bounded by the contours L_0 and L_1. On one of the contours, for example, on L_0, we introduce an auxiliary function $\omega(t)$, which is defined in the following

way:

$$2\omega(t) = \varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}. \tag{3.31}$$

Then, from (3.11), we get the following equalities:

$$\begin{aligned} \varphi(t) &= \omega(t), \\ \psi(t) &= -\overline{\omega(t)} - \bar{t}\omega'(t). \end{aligned} \tag{3.32}$$

Let us introduce a Cauchy-type integral given by $\dfrac{1}{2\pi i}\displaystyle\int_{L_0} \frac{\omega(t)}{t-z}\,dt$. Then, in accordance with the Sokhotskii-Plemelj formula (1.14′), Ch. 1, Vol. 1, the function $\omega(t)$ may be represented in the form

$$\varphi(t) - \lim_{\substack{z\to t\\ z\in D_0^+}} \frac{1}{2\pi i}\int_{L_0} \frac{\omega(t)}{t-z}\,dt = -\lim_{\substack{z\to t\\ z\in D_0^-}} \frac{1}{2\pi i}\int_{L_0} \frac{\omega(t)}{t-z}\,dt. \tag{3.33}$$

The function $\varphi(t)$ is the boundary value of a function which is analytical in the domain D, the function $\dfrac{1}{2\pi i}\displaystyle\int_{L_0} \frac{\omega(t)}{t-z}\,dt$ $(z \in D_0^+)$ is the boundary value of a function which is analytical in the domain D_0^+, while the function $\dfrac{1}{2\pi i}\displaystyle\int_{L_0} \frac{\omega(t)}{t-z}\,dt$ $(z \in D_0^-)$ is the boundary value of a function which is analytical in the domain D_0^-. Thus, the expression on the left-hand side of (3.33) is the boundary value of a function which is analytical in D, while the expression on the right-hand side is the boundary value of a function which is analytical in D_0. Since their limiting values are identical, they may be considered to represent a single function, which is analytical in the domain $D_1^- = D \cup D_0^-$, and which will henceforth be denoted by $\varphi^*(z)$. Thus, we have

$$\varphi^*(z) = \varphi(z) - \frac{1}{2\pi i}\int_{L_0} \frac{\omega(t)}{t-z}\,dt = -\frac{1}{2\pi i}\int_{L_0} \frac{\omega(t)}{t-z}\,dt. \tag{3.34}$$

It should be noted that the expressions for the integrals on the left- and right-hand sides are apparently identical, but the integrals themselves are different, since they are considered in different domains ($z \in D$ on the left-hand side, and $z \in D_0^-$ on the right).

By performing similar constructions for the functions $\psi(z)$, we arrive at a new function $\psi^*(z)$, which is analytical in the domain D_1^-, and which can be represented

as follows in each of the domains D and D_0^-:

$$\psi^*(z) = \psi(z) + \frac{1}{2\pi i}\int_{L_0}\frac{\overline{\omega(t)} + \bar{t}\omega'(t)}{t - z}dt = \frac{1}{2\pi i}\int_{L_0}\frac{\overline{\omega(t)} + \bar{t}\omega'(t)}{t - z}dt. \quad (3.35)$$

Since the functions $\varphi^*(z)$ and $\psi^*(z)$ are analytical in the domain D_1^-, the problem of their determination (under the assumption that $\omega(t)$ is conditionally specified) is reduced to the solution of the problem of the theory of elasticity for this domain. The boundary condition can be easily obtained with the help of Eqs. (3.34) and (3.35), but an explicit expression for it is quite cumbersome, and hence we can write this condition in the symbolic notation

$$\varphi^*(t) + t\overline{\varphi^{*\prime}(t)} + \overline{\psi^*(t)} = f_1(t) + c + H(\omega, t). \quad (3.36)$$

Suppose that $\varphi^*(z) = H_1[\omega(t), z]$ and $\psi^*(z) = H_2[\omega(t), z]$ is the solution of the problem (3.36). Then, with the help of Eq. (3.31), we get a Fredholm equation of the second kind for $\omega(t)$:

$$\omega(t_0) = \frac{1}{2\pi i}\int_{L_0}\omega(t)d\ln\frac{t - t_0}{\bar{t} - \bar{t}_0} + \frac{1}{2\pi i}\int_{L_0}\overline{\omega(t)}d\frac{t - t_0}{\bar{t} - \bar{t}_0}$$

$$+ \left\{H_1[\omega(t), t_0] - t_0\frac{\partial H_1\overline{[\omega(t), t_0]}}{\partial t_0} - H_2[\overline{\omega(t), t_0}]\right\} + H_3[\omega(t), t_0], \quad (3.37)$$

where $H_3[\omega(t), t_0]$ denotes the terms defined by the additional terms in (3.34) and (3.35). The algorithm concludes with the solution of Eq. (3.37).

It should be noted that the introduction of the function $\omega(t)$ helps in solving the problem when mixed type of conditions are given[3] on the contour L_1, as well as in the case when the contour L_1 degenerates into a cut [26].

In case the contour on which the auxiliary function is introduced happens to be a circle, it is convenient to start with the Fourier series expansion of the function $\omega(t)$ in order to solve Eq. (3.37). It has been established [27] that the system of algebraic equations obtained in this case is quasi-regular. The proof of this statement has been obtained under the assumption that the conformal mapping of the exterior of the contour onto a circle may be represented in the form of a rational function.

Section 4
Application of Cauchy-type Integrals. Series Solutions

It was shown in Sec. 1 that the problems of flexure and bending of bars, which can be reduced to harmonic problems, can be readily solved, in principle, by using

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[3] Naturally, we can speak of only such cases, where the solution of the mixed problem can be effectively realized for the domain D_1^-.

the method of conformal mapping. The solution can be obtained in the form of a certain integral (Schwarz's integral), and can be constructed in an explicit form if the mapping function is rational. The problem is much more complicated when we consider a plane problem or the problem of bending of plates, which are reduced to a biharmonic problem. The application of conformal mapping can lead to effective results only when the mapping function is rational-fractional. For the sake of simplicity, we shall confine ourselves to the case when the mapping function is rational.

By way of an example, let us consider the internal problem in the theory of elasticity for the domain D^+ which is bounded by the contour L. For the sake of convenience of analysis, we rewrite the first and second problems (2.22) and (2.23) in a unified form

$$k\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t), \tag{4.1}$$

where $k = -\varkappa$ for the first problem, while in the second problem, $k = 1$. In what follows, we shall be frequently considering the second problem (on account of its importance from the application point of view) with certain additional conditions required for self-balancing of external forces.

Suppose that the function $z = \omega(\zeta)$, which maps a unit circle in the auxiliary plane ζ onto the domain D^+, has the form (for the sake of simplicity, the zero-order term has been omitted)

$$\omega(\zeta) = c_1\zeta + c_2\zeta^2 + \ldots + c_n\zeta^n. \tag{4.2}$$

We go over to the new functions $\varphi[\omega(\zeta)]$ and $\psi[\omega(\zeta)]$, which we shall be denoting by $\varphi(\zeta)$ and $\psi(\zeta)$. The boundary condition (4.1) can then be rewritten in the following form:

$$k\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f(\sigma). \tag{4.3}$$

Following [17], we act on both sides of this equation by the Cauchy operator for $|\zeta| < 1$. Since the function $\varphi(\zeta)$ is analytical in the domain $|\zeta| < 1$, we find from the Cauchy integral theorem that the integral of the first term reconstructs the function $\varphi(\zeta)$.

Let us now consider the third term. The function $\overline{\psi(\sigma)}$ is conjugate to $\psi(\sigma)$ which is the boundary value of a function analytical in the domain D^+. We shall show that $\overline{\psi(\sigma)}$ is the boundary value of a function analytical in the domain D^-. From this it directly follows that the integral vanishes.

In order to prove this, we represent the function $\psi(\zeta)$ in the form of a series

$$\psi(\zeta) = \sum_{n=1}^{\infty} a_n\zeta^n. \tag{4.4}$$

Let us now consider the series

$$\sum_{n=1}^{\infty} \bar{a}_n \frac{1}{\zeta^n} = \bar{\psi}\left(\frac{1}{\zeta}\right),$$

which is an analytical function in the domain D^-. The limiting value of this function on the circle $|\zeta| = 1$ coincides with $\overline{\psi(\sigma)}$, Q.E.D.

We now go over to the second term. The ratio $\omega(\sigma)/\overline{\omega'(\sigma)}$ is the boundary value of the function $\omega(\zeta)/\bar{\omega}'(1/\zeta)$, which is analytical in the domain $|\zeta| > 1$ if there is a pole of order n at infinity. Hence this ratio can be represented in the form of a series

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)} = b_n\zeta^n + b_{n-1}\zeta^{n-1} + \ldots + b_1\zeta + b_0 + \sum_{k=1}^{\infty} b_{-k}\frac{1}{\zeta^k}. \tag{4.5}$$

The coefficients b_n can easily be determined (it will be shown later that in order to solve the problem, it is necessary to determine only the first $n + 1$ coefficients).

In a similar way, we can show that the function $\overline{\varphi'(\sigma)}$ is the boundary value of the function $\bar{\varphi}'(1/\zeta)$, which is analytical in the domain $|\zeta| > 1$.

Thus, the second term in (4.3) is the boundary value of the product

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\bar{\varphi}'\left(\frac{1}{\zeta}\right).$$

Using the series representation of the function $\varphi(\zeta)$,

$$\varphi(\zeta) = a_0 + a_1\zeta + \ldots + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \ldots \qquad (|\zeta| < 1), \tag{4.6}$$

we get the main part of the product mentioned above:

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\bar{\varphi}'\left(\frac{1}{\zeta}\right) = K_n\zeta^n + K_{n-1}\zeta^{n-1} + \ldots + K_1\zeta + K_0, \tag{4.7}$$

where the coefficients K_j are defined as follows:

$$\begin{aligned}
K_n &= \bar{a}_1 b_n,\\
K_{n-1} &= \bar{a}_1 b_{n-1} + 2\bar{a}_2 b_n,\\
K_{n-2} &= \bar{a}_1 b_{n-2} + 2\bar{a}_2 b_{n-1} + 3\bar{a}_3 b_n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
K_1 &= \bar{a}_1 b_1 + 2\bar{a}_2 b_2 + \ldots + (n-1)\,\bar{a}_{n-1} + n\bar{a}_n b_n,\\
K_0 &= \bar{a}_1 b_0 + 2\bar{a}_2 b_1 + \ldots + n\bar{a}_n b_{n-1} + (n+1)\,\bar{a}_{n+1} b_n.
\end{aligned} \tag{4.8}$$

If we act on the second term by the Cauchy operator, we obtain a finite sum (the

term K_0 has been omitted, since it does not affect the state of stress)

$$K_n\zeta^n + K_{n-1}\zeta^{n-1} + \dots + K_2\zeta^2 + K_1\zeta = \frac{1}{2\pi i}\int\limits_{|\sigma|=1} \frac{\omega(\sigma)\overline{\varphi'(\sigma)}d\sigma}{\overline{\omega'(\sigma)}(\sigma-\zeta)}, \tag{4.9}$$

since, in accordance with (4.7), the remaining terms in the expression for the density represent the boundary value of a function which is analytical in the domain $|\zeta| > 1$. Of course, the constants $K_n, K_{n-1}, \dots, K_2, K_1$ are as yet unknown.

Summing up all the results obtained hitherto, we get the equality

$$k\varphi(\zeta) + K_n\zeta^n + K_{n-1}\zeta^{n-1} + \dots + K_2\zeta^2 + K_1\zeta$$
$$= \frac{1}{2\pi i}\int\limits_{|\sigma|=1} \frac{f\,d\sigma}{\sigma-\zeta} = A(\zeta). \tag{4.10}$$

We shall use this equation for determining the constants K_j which, in accordance with (4.8), can be expressed in terms of the coefficients a_j. We expand the right-hand side of (4.10) into a power series in ζ (for $|\zeta| < 1$). For this purpose, we represent the function $A(\zeta)$ in the following form:

$$\frac{1}{2\pi i}\int\limits_{|\sigma|=1} \frac{f(\sigma)\,d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int\limits_{|\sigma|=1} \frac{f(\sigma)\,d\sigma}{\sigma\left(1-\dfrac{\zeta}{\sigma}\right)}$$

$$= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{f(\sigma)}{\sigma}\left[\sum_{i=0}^{\infty}\left(\frac{\zeta}{\sigma}\right)^i\right]d\sigma = \sum_{i=0}^{\infty} A_i\zeta^i.$$

It is obvious that the coefficients A_i are given by

$$A_i = \frac{1}{2\pi i}\int\limits_{|\sigma|=1} f(\sigma)\sigma^{-i-1}d\sigma \qquad (i = 1, 2, \dots, n). \tag{4.11}$$

Next, having obtained the expansion of both sides of Eq. (4.10) into series in ζ, we equate the coefficients of like powers (up to n inclusively). This leads to the following system of equations:

$$\begin{aligned}
&a_n + \bar{a}_1 b_n = A_n,\\
&a_{n-1} + \bar{a}_1 b_{n-1} + 2\bar{a}_2 b_n = A_{n-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_2 + \bar{a}_1 b_2 + 2\bar{a}_2 b_3 + \dots + (n-1)\bar{a}_{n-1}b_n = A_2,\\
&a_1 + \bar{a}_1 b_1 + 2\bar{a}_2 b_2 + \dots + (n-1)\bar{a}_{n-1}b_{n-1} + n\bar{a}_n b_n = A_1.
\end{aligned} \tag{4.12}$$

Generally speaking, this system is degenerate since it must be unsolvable for any

right-hand side. The condition for its solvability is that the principal moment of external forces be equal to zero[4], and it must be satisfied as per formulation of the boundary value problem. On the other hand, it has been mentioned earlier that the state of stress does not depend on $\mathrm{Im}\, a_1$ and can therefore be arbitrarily fixed. As a result, we arrive at a system which can be solved only uniquely.

By finding the coefficients a_j, we can determine the coefficients K_j. This enables us to treat Eq. (4.10) as an exact solution for the function $\varphi(\zeta)$ (after transferring all the terms with coefficients K_j to the right-hand side).

In order to complete the solution of the problem, we must also determine the function $\psi(\zeta)$, and for this purpose we proceed as follows. We consider the boundary condition (4.3), take its conjugate, and apply the Cauchy operator to it. Then, repeating the above digression with slight alterations, we arrive at the equality[5]

$$\psi(\zeta) = \frac{1}{2\pi i} \int\limits_{|\sigma|=1} \frac{\bar{f}\, d\sigma}{\sigma - \zeta} + \bar{K}_0 + \bar{K}_{-1}\zeta + \bar{K}_{-2}\zeta^2 + \dots . \tag{4.13}$$

It is possible to find the sum of the series appearing in this equation. As a matter of fact, a direct expansion into series shows that the following equation holds:

$$\bar{K}_0 + \bar{K}_{-1}\zeta + \bar{K}_{-2}\zeta^2 + \dots = -\frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\varphi'(\zeta) + \frac{\bar{K}_1}{\zeta} + \dots + \frac{\bar{K}_n}{\zeta^n} - \bar{a}_0.$$

Then, we get the following expression for $\psi(\zeta)$:

$$\psi(\zeta) = \frac{1}{2\pi i} \int\limits_{|\sigma|=1} \frac{\bar{f}\, d\sigma}{\sigma - \zeta} - \frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\varphi'(\zeta) + \frac{\bar{K}_1}{\zeta} + \dots + \frac{\bar{K}_n}{\zeta^n} - \bar{a}_0. \tag{4.14}$$

The method described above can be extended, almost without any alterations, to the case of an external problem, when the mapping function is represented by the expression

$$\omega(\zeta) = \frac{c_{-1}}{\zeta} + c_1\zeta + \dots + c_n\zeta^n \tag{4.15}$$

(the existence of other terms containing negative powers is ruled out in view of the conformity of mapping at infinity). In this case, it is necessary, of course, that when the Cauchy operator is applied, the point ζ must lie in the domain $|\zeta| > 1$ (the examples given at the end of this section visually demonstrate this).

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[4] This will be illustrated once again on the example of the simplest problem, when the domain D^+ is a circle.

[5] The coefficients K_j ($j < 0$) are introduced in the same way as in (4.7).

It should be noted that by making the reasoning more complicated, a solution can be obtained for the case when the mapping function is rational-fractional.

Before considering the question of the application of conformal mapping for solving problems in the theory of elasticity for semi-infinite domains (i.e. for domains bounded by open contours), let us make some preliminary remarks about the admissible configuration of boundaries and the constraints on the boundary conditions.

Let L be an open contour extending at both ends to infinity, and bifurcating a plane into domains D^+ and D^-. The positive direction is taken as the one for which the domain D^- is situated to the left. We fix a certain point q and take two points a and b on the line L. We consider the behaviour of the rays qa and qb as the points a and b go off to infinity in opposite directions, and require that limiting positions for these rays exist for any arbitrarily chosen point q. The angle between these limiting rays will be denoted by Π. It can be easily seen that this angle will be the same for all points lying on the same side of the contour L, and that the following equality is obviously satisfied:

$$\Pi(q) - \Pi(q') = 2\pi, \tag{4.16}$$

where the points q and q' are situated in the domains D^+ and D^- respectively.

We shall preset the conditions for the behaviour of the functions $\Phi(z)$ and $\Psi(z)$ at infinity:

$$\Phi(z) = \frac{\gamma}{z} + o\left(\frac{1}{z}\right), \qquad \Psi(z) = \frac{\gamma'}{z} + o\left(\frac{1}{z}\right) \tag{4.17}$$

or

$$\varphi(z) = \gamma \ln z + o(1), \qquad \psi(z) = \gamma' \ln z + o(1), \tag{4.17'}$$

where γ and γ' are arbitrary complex constants.

We now require that the magnitude of the principal vector of external forces be limited, and write down its expression in accordance with (2.9) and (4.16):

$$[P_x + iP_y]_a^b = \gamma \ln \frac{r''}{r'} + \gamma\Pi' i + \gamma(e^{2i\beta} - e^{2i\alpha}) + \bar{\gamma}' \ln \frac{r''}{r'} - \bar{\gamma}'\Pi' i + \varepsilon. \tag{4.18}$$

Here, α and β denote the angles formed by the rays qa and qb with the x-axis (the point q is chosen arbitrarily in D^+), r' and r'' are the distances from the point q to the points a and b, Π' is the angle, analogous to Π, for an arbitrary point in the domain D^-, and ε is a small quantity which tends to zero as the points a and b go off to infinity. In order that the magnitude of the principal vector be limited (for any position of the point q and hence for any values of r' and r''), it is necessary that the equality $\gamma + \bar{\gamma}' = 0$ be satisfied. Then, in the limit, we get from (4.18)

$$P_x + iP_y = 2\Pi'\gamma + i(e^{2i\beta^*} - e^{2i\alpha^*})\bar{\gamma}, \tag{4.19}$$

where α^* and β^* are the limiting values of the angles α and β. It is obvious that $\beta^* - \alpha^* = \Pi'$. The quantity γ can be represented in terms of P_x and P_y as follows:

$$\gamma = \frac{2\Pi'(P_x + iP_y) + i(e^{2i\beta^*} - e^{2i\alpha^*})(P_x - iP_y)}{4[(\Pi')^2 - \sin^2\Pi']}. \tag{4.20}$$

Apparently, for $\Pi' = 0$, we must require additionally that $P_x = P_y = 0$.

We conclude our discussion of the general questions concerning the solution of problems for semi-infinite domains (in view of the analogy with the discussion at the beginning of this section), and go over to specific examples.

Suppose that the domain occupied by an elastic medium is situated outside the parabola $x^2 = 4a^2(y + a^2)$ $(a > 0)$. In this case, the function that performs a conformal mapping onto a half-plane has the form

$$z = \omega(\zeta) = i(\zeta - ia)^2. \tag{4.21}$$

Suppose that we go around from left to right (D^- is the exterior of the parabola). In this case, the angle $\Pi' = -2\pi$.

Unlike the method considered earlier, the boundary condition in this case can be conveniently written not in the form (4.1), but directly in terms of stresses (see (2.8″)). The boundary value problem for the functions $\Phi[\omega(\zeta)] = \Phi(\zeta)$ and $\Psi[\omega(\zeta)] = \Psi(\zeta)$ can be formulated as follows:

$$\Phi(\sigma) + \overline{\Phi(\sigma)} + \frac{(\sigma + ia)}{2}\Phi'(\sigma) - \frac{(\sigma - ia)}{(\sigma + ia)}\Psi(\sigma) = N_\nu + iT_\nu, \tag{4.22}$$

where the functions N_ν and T_ν are, as usual, the projections of the stress vector onto the normal and the tangent to the contour (i.e. the projections in coordinates defined by the conformal mapping).

Multiplying Eq. (4.22) by $\sigma + ia$ and taking its conjugate value, we can transform this equation as follows:

$$(\sigma - ia)\Phi(\sigma) + (\sigma - ia)\overline{\Phi(\sigma)} + \frac{(\sigma - ia)^2}{2}\overline{\Phi'(\sigma)} - (\sigma + ia)\overline{\Psi(\sigma)} = (\sigma - ia)(N_\nu - iT_\nu). \tag{4.23}$$

In view of the conditions (4.17), the first term is the boundary value of the function $(\zeta - ia)\Phi(\zeta)$ which is analytical for $\text{Im}\,\zeta < 0$, including the point at infinity. The second, third, and fourth terms are the boundary values of the functions $(\zeta - ia)\overline{\Phi}(\zeta)$, $(\zeta - ia)^2\overline{\Phi}'(\zeta)$ and $(\zeta + ia)\overline{\Psi}(\zeta)$ which are analytical for $\text{Im}\,\zeta > 0$, including the point at infinity. Hence, applying the Cauchy operator $\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\sigma}{\sigma - \zeta}$ $(\text{Im}\,\zeta < 0)$ to the left- and right-hand sides of (4.23), we directly obtain the required representation for the function $\Phi(\zeta)$:

$$\Phi(\zeta) = -\frac{1}{2\pi i(\zeta - ia)} \int_{-\infty}^{+\infty} \frac{(\sigma - ia)(N_\nu - iT_\nu)}{\sigma - \zeta} d\sigma. \tag{4.24}$$

Applying the Cauchy operator to the condition conjugate to (4.23), we get an expression for the function $\Psi(\zeta)$:

$$\Psi(\zeta) = \frac{1}{2\pi i(\zeta - ia)} \int_{-\infty}^{+\infty} \frac{(\sigma + ia)(N_\nu + iT_\nu)}{\sigma - \zeta} d\sigma$$

$$+ \frac{\zeta + ia}{\zeta - ia}\Phi(\zeta) + \frac{(\zeta + ia)^2}{2(\zeta - ia)}\Phi'(\zeta). \tag{4.25}$$

These formally obtained equations directly lead to additional constraints which must be imposed[6] on the functions N_ν and T_ν.

Let us now go over to a consideration of the mixed problem. Suppose that on the contour L bounding a domain we are given a sequence of points α_k and β_k $(k = 1, 2, \ldots, n)$ which split the contour into separate arcs γ_k (whose ends are given by the points α_k and β_k) and γ_k^* (whose ends are given by the points β_k and α_{k+1}). We denote the system of arcs γ_k by L_1, and the arcs γ_k^* by L_2 $(L = L_1 \cup L_2)$. We preset the conditions of mixed type on the contour L:

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = f_k(t) \quad (t \in \gamma_k), \tag{4.26}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f_k^*(t) + c_k \quad (t \in \gamma_k^*). \tag{4.27}$$

The functions f_k and f_k^* are assumed to be given.

In order to solve the mixed problem in the theory of elasticity, we use the following representations [28] for the functions $\varphi(z)$ and $\psi(z)$:

$$\varphi(z) = \frac{1}{2\pi i} \int_L \frac{\omega(t)dt}{t - z}, \tag{4.28}$$

$$\psi(z) = -\frac{\varkappa}{2\pi i} \int_L \frac{\overline{\omega(t)}dt}{t - z} + \frac{1}{2\pi i} \int_L \frac{\omega(t)d\bar{t}}{t - z} - \frac{1}{2\pi i} \int_L \frac{\bar{t}\omega(t)dt}{(t - z)^2}. \tag{4.29}$$

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[6] These constraints emerge from the conditions of the existence of integrals in (4.24) and (4.25) and consist in the requirement that the functions $F = N_\nu + iT_\nu$ and F' satisfy the H-L conditions. Besides, σF and $\sigma^2 F'$ satisfy the H-L conditions in the neighbourhood of an infinitely remote point.

With the help of formulas (1.14), Ch. 1, Vol. 1, we obtain the limiting values of $\varphi^+(t)$, $\varphi'^+(t)$ and $\psi^+(t)$. Substituting these values into the boundary conditions (4.26) and (4.27), we obtain a singular integral equation

$$A(t_0)\omega(t_0) + \frac{B(t_0)}{\pi i}\int_L \frac{\omega(t)}{t - t_0}dt + \int_L k_1(t_0, t)\omega(t)dt + \int_L \overline{k_2(t_0, t)}\,\overline{\omega(t)}\,\overline{dt} = f(t_0) + c(t_0), \quad (4.30)$$

where

$$k_1(t_0, t) = \frac{\varkappa}{2\pi i}\frac{\partial}{\partial t}\ln\frac{\bar{t} - \bar{t}_0}{t - t_0}, \qquad k_2(t_0, t) = \frac{1}{2\pi i}\frac{\partial}{\partial t}\frac{\bar{t} - \bar{t}_0}{t - t_0},$$

$$\{A(t_0), B(t_0), c(t_0)\} = \begin{cases} \dfrac{1-\varkappa}{2}, & \dfrac{1+\varkappa}{2}, \quad c_k \quad (t_0 \in L_1), \\ -\varkappa, & 0, \quad 0 \quad (t_0 \in L_2). \end{cases}$$

Since Eq. (4.30) includes a term containing $\overline{\omega(t)}$, the solvability condition for this equation has the form (see (3.14), Ch. 1, Vol. 1)

$$\mathrm{Re}\int_L [f(t) + c(t)]\sigma_k(t)dt = 0 \quad (t = 1, 2, \ldots, n),$$

where $\sigma_k(t)$ is the complete system of companion solutions of the companion equation. In this case, the solutions are bounded at all points (see Sec. 3, Ch. 1, Vol. 1). It should be noted that an analysis of Eq. (4.30) has been given in [29].

By way of an example, let us consider some specific problems.

Suppose that the original domain is a circle of radius R. The mapping function in this case is extremely simple[7]: $z = \omega(\zeta) = R\zeta$. Since the index n in (4.2) is equal to unity, the application of the Cauchy operator to the left-hand side of the boundary condition gives the following expression for the left-hand side of (4.10):

$$\varphi(\zeta) + \bar{a}_1\zeta + 2\bar{a}_2 = \frac{1}{2\pi i}\int_{|\sigma|=1}\frac{f\,d\sigma}{\sigma - \zeta} = \frac{1}{2\pi i}\int_{|\sigma|=1}\frac{f\,d\sigma}{\sigma} + \frac{\zeta}{2\pi i}\int_{|\sigma|=1}\frac{f\,d\sigma}{\sigma^2} + \ldots. \quad (4.31)$$

[7] Strictly speaking, there is no need to introduce the conformal mapping in this case. It is better to directly apply the Cauchy operator to the boundary conditions. However, for the sake of uniformity of notation, the previous approach has been retained in this case.

On the other hand, we obtain the following expression for $\psi(\zeta)$ in accordance with (4.14):

$$\psi(\zeta) = \frac{1}{2\pi i} \int\limits_{|\sigma|=1} \frac{\bar{f}\, d\sigma}{\sigma - \zeta} - \frac{\varphi'(\zeta)}{\zeta} + \frac{a_1}{\zeta} - \overline{\varphi(0)}. \qquad (4.32)$$

With the help of the representation (4.31), we can directly establish the conditions for the solvability of a given problem and provide a visual interpretation of these conditions. Indeed, let us compare the coefficients of the first-order terms on the left- and right-hand sides. This gives

$$a_1 + \bar{a}_1 = \frac{1}{2\pi i} \int\limits_{|\sigma|=1} \frac{f\, d\sigma}{\sigma^2}. \qquad (4.33)$$

Consequently, there must be a real number on the right-hand side. Introducing the polar angle θ, we rewrite the right-hand side of Eq. (4.33):

$$\frac{1}{2\pi} \int\limits_0^{2\pi} (f_1 + if_2) e^{-i\theta} d\theta$$

$$= \frac{1}{2\pi} \int\limits_0^{2\pi} (f_1 \cos\theta + f_2 \sin\theta)\, d\theta + \frac{i}{2\pi} \int\limits_0^{2\pi} (-f_1 \sin\theta + f_2 \cos\theta)\, d\theta.$$

It can be easily shown that the equality to zero of the second term is equivalent to the condition that the principal vector moment of external forces be equal to zero (we shall discuss this in detail at a later stage). We shall make a remark of a general nature. While directly calculating an integral, we may find that the required condition is violated on account of an error introduced by a quadrature formula. Generally speaking, this may result in the non-solvability of Eq. (4.33). Naturally, the imaginary part on the right-hand side must be small and should be neglected.

Let us now specify the boundary conditions. Suppose that concentrated forces $P^1_{x\nu} + iP^1_{y\nu}$, $P^2_{x\nu} + iP^2_{y\nu}$, ... , $P^n_{x\nu} + iP^n_{y\nu}$ are applied at the points $z_1 = Re^{i\alpha_1}$, $z_2 = Re^{i\alpha_2}$, ... , $z_n = Re^{i\alpha_n}$. Then in accordance with (2.10), the function $f(t)$ is explicitly defined in the form of a piecewise analytical function. For the sake of definiteness, we shall assume that the function $f(t) = 0$ on the arc $\alpha_n \alpha_1$. Then,

$$f(t) = i(P^1_{x\nu} + iP^1_{y\nu}) \text{ on } \alpha_1,\ \alpha_2,$$

$$f(t) = i(P^1_{x\nu} + iP^1_{y\nu}) + i(P^2_{x\nu} + iP^2_{y\nu}) \text{ on } \alpha_1, \alpha_3, \text{ etc.}$$

From the equilibrium condition for the body as a whole, it follows that upon returning to the arc α_n, α_1, we must get the zero value again.

We can write

$$\frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{f\,d\sigma}{\sigma-\zeta}=-\frac{1}{2\pi}\{(P^1_{x\nu}+iP^1_{y\nu})\ln(\sigma_1-\zeta)$$

$$+(P^2_{x\nu}+iP^2_{y\nu})\ln(\sigma_2-\zeta)+\ldots+(P^n_{x\nu}+iP^n_{y\nu})\ln(\sigma_n-\zeta)\}. \quad (4.34)$$

As in the previous case, we have

$$\frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{\bar{f}\,d\sigma}{\sigma-\zeta}=\frac{1}{2\pi}\{(P^1_{x\nu}-iP^1_{y\nu})\ln(\sigma_1-\zeta)$$

$$+(P^2_{x\nu}-iP^2_{y\nu})\ln(\sigma_2-\zeta)+\ldots+(P^n_{x\nu}-iP^n_{y\nu})\ln(\sigma_n-\zeta)\}. \quad (4.35)$$

Let us also write the expression for the coefficient a_1:

$$a_1=\frac{1}{4\pi}\sum_{k=1}^{n}\frac{P^k_{x\nu}+iP^k_{y\nu}}{\sigma_k}\qquad(z_k=R\sigma_k).$$

Then, in accordance with (4.31) and (4.32), we get

$$\varphi(\zeta)=-\frac{1}{2\pi}\sum_{k=1}^{n}(P^k_{x\nu}+iP^k_{y\nu})\ln(\sigma_k-\zeta)$$

$$-\frac{\zeta}{4\pi}\sum_{k=1}^{n}(P^k_{x\nu}+iP^k_{y\nu})\bar{\sigma}_k, \quad (4.36)$$

$$\psi(\zeta)=\frac{1}{2\pi}\sum_{k=1}^{n}(P^k_{x\nu}-iP^k_{y\nu})\ln(\sigma_k-\zeta)$$

$$-\frac{1}{2\pi}\sum_{k=1}^{n}\frac{P^k_{x\nu}+iP^k_{y\nu}}{\sigma_k-\zeta}\bar{\sigma}_k. \quad (4.37)$$

Finally, let us write the formulas for the functions $\varphi(z)$ and $\psi(z)$ as well as for stresses in the case when the forces $(P, 0)$ and $(-P, 0)$ are applied at the points $z_1 = Re^{i\alpha}$ and $z_2 = Re^{i(\pi-\alpha)}$ (Fig. 34):

$$\varphi(z)=-\frac{P}{2\pi}\left[\ln(z_1-z)-\ln(z_2-z)+\frac{\bar{z}_1-\bar{z}_2}{2R^2}z\right],$$

$$\psi(z)=\frac{P}{2\pi}\left[\ln(z_1-z)-\ln(z_2-z)-\frac{\bar{z}_1}{z_1-z}+\frac{\bar{z}_2}{z_2-z}\right]. \quad (4.38)$$

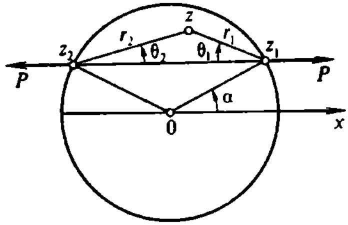

Fig. 34. A circle subjected to two concentrated forces.

It is convenient to introduce the local polar angles θ_1 and θ_2 for representing the stress components (the angles are considered positive when the point z is situated above the line of action of forces):

$$\sigma_x = \frac{2P}{\pi}\left[\frac{\cos^3\theta_1}{r_1} + \frac{\cos^3\theta_2}{r_2}\right] - \frac{P}{\pi R}\cos\alpha,$$
$$\sigma_y = \frac{2P}{\pi}\left[\frac{\sin^2\theta_1\cos\theta_1}{r_1} + \frac{\sin^2\theta_2\cos\theta_2}{r_2}\right] - \frac{P}{\pi R}\cos\alpha, \tag{4.39}$$
$$\tau_{xy} = -\frac{2P}{\pi}\left[\frac{\sin\theta_1\cos^2\theta_1}{r_1} - \frac{\cos\theta_2\cos^2\theta_2}{r_2}\right].$$

It is interesting to analyze the case when $\alpha = 0$ ($z_1 = R$ and $z_2 = -R$). Let us consider the expression for the normal component σ_y on the real axis, i.e. for $\theta_1 = \theta_2 = 0$. Since the angle $\alpha = 0$, we get $\sigma_y = -P/\pi R$. However, this is valid only for internal points, since $r_1 = 0$ at the point z_1, and $r_2 = 0$ at the point z_2. Consequently, the stresses are unlimited at these points. Indeed, let us make a cut along the diameter $y = 0$. If the stresses at these points were limited, we would arrive at a contradiction, since the solution thus obtained must correspond to the problem for a semicircle which is loaded along the diameter by the pressure $P/\pi R$, and at the corner points by the concentrated forces $\pm P/2$. The projection of forces onto the vertical axis is equal to $2P/\pi$ and is hence non-zero. To attain balance, forces P/π must be applied at the corner points. This is in complete accord with the concepts of concentrated forces (see Sec. 6, Ch. 3, Vol. 1), when the solution may be considered as the limiting solution for the case of a domain containing a depression of decreasing size.

Let us consider another problem. Suppose that we have a plane with an elliptical hole, whose semi-axes are equal to a and b. The mapping function has the form

$$z = \omega(\zeta) = \frac{a+b}{2}\left(\zeta + \frac{a-b}{a+b}\frac{1}{\zeta}\right) = A\left(\zeta + \frac{m}{\zeta}\right) \tag{4.40}$$

$$(A > 0,\ 0 \leqslant m \leqslant 1).$$

Upon a transition to the auxiliary variable σ, the boundary condition can be written as

$$\varphi(\sigma) + \frac{1}{\sigma}\,\frac{\sigma^2 + m}{1 - m\sigma^2}\,\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f(\sigma). \tag{4.41}$$

Applying the Cauchy operator, we directly get (ζ is inside the domain)

$$\frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{\varphi(\sigma)}{\sigma - \zeta}d\sigma = -\varphi(\sigma), \frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{\overline{\psi(\sigma)}}{\sigma - \zeta}d\sigma = 0. \tag{4.42}$$

The change in sign has been caused due to the fact that we have made a round in the anticlockwise direction.

We consider the second term in the condition (4.41) and represent it in the form of the boundary value of the function $\frac{1}{\zeta}\,\frac{\zeta^2 + m}{1 - m\zeta^2}\bar{\varphi}'\left(\frac{1}{\zeta}\right)$ which is (when $0 \leqslant m \leqslant 1$) an analytical function in the domain $|\zeta| < 1$. Therefore, the Cauchy operator applied to the second term cancels it. Turning to Eqs. (4.42), we immediately obtain an expression for the function $\varphi(\zeta)$:

$$\varphi(\zeta) = -\frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{f(\sigma)}{\sigma - \zeta}\,d\sigma. \tag{4.43}$$

Considering the condition conjugate to (4.3) and applying the Cauchy operator once again, we obtain an expression for the function $\psi(\zeta)$:

$$\psi(\zeta) = -\frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{\bar{f}d\sigma}{\sigma - \zeta} - \zeta\,\frac{1 + m\zeta^2}{\zeta^2 - m}\,\varphi'(\zeta). \tag{4.44}$$

Let us now specify the boundary condition, considering that the load consists only of the hydrostatic pressure p. Consequently, in accordance with (2.25), we get

$$f(t) = -pt = -pA\left(\sigma + \frac{m}{\sigma}\right). \tag{4.45}$$

Taking this into account, we obtain

$$\frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{f(\sigma)}{\sigma - \zeta}d\sigma = -\frac{pAm}{\zeta}, \quad \frac{1}{2\pi i}\int\limits_{|\sigma|=1}\frac{\overline{f(\sigma)}}{\sigma - \zeta}d\sigma = -\frac{pA}{\zeta}.$$

The final expressions for the functions $\varphi(\zeta)$ and $\psi(\zeta)$ have the form

$$\varphi(\zeta) = -\frac{pAm}{\zeta}, \tag{4.46}$$

$$\psi(\zeta) = -\frac{pA}{\zeta} - \frac{pAm}{\zeta}\,\frac{1 + m\zeta^2}{\zeta^2 - m}. \tag{4.47}$$

Let us now consider a limiting case when the ellipse degenerates into a cut ($m = 1, A = a/2$). In this case the stresses σ_x and σ_y on the positive part of the real axis are given by

$$\sigma_x = \sigma_y = \frac{2p}{\rho^2 - 1} \tag{4.48}$$

or

$$\sigma_x = \sigma_y = \frac{p}{2}\,\frac{\sqrt{a}}{\sqrt{x-a}} + M(x) = \frac{K}{\sqrt{x-a}} + M(x), \tag{4.48$'$}$$

where the parameter ρ is connected with the coordinate x through the relation $x = A(\rho + 1/\rho)$ and $M(x)$ is a bounded function. It follows from (4.48) that as we approach the end of the cut, these stresses become unlimited.

The results obtained above directly lead to the solution of two more problems: (1) when a hydrostatic pressure p is applied at infinity, and (2) when there is a uniaxial stress (along the y-axis). In the first case we must superimpose on the required solution a solution which has the form $\sigma_x^0 = \sigma_y^0 = p$, $\tau_{xy}^0 = 0$ in terms of stresses. This leads to the problem considered above. If, on the other hand, we subtract from the solution thus obtained a solution for which $\sigma_x^0 = p$ and $\sigma_y^0 = \tau_{xy}^0 = 0$, we arrive at the second problem.

A characteristic feature of all the three problems is that the solution depends on the same value of the coefficient K, which is called the stress intensity factor. As we have already mentioned in Sec. 9, Ch. 3, Vol. 1, this coefficient plays an extremely important role in the theory of fracture.

We shall now describe a method for obtaining series solutions of the problems in the theory of elasticity. This method is directly applied to the domains bounded by a circle or two concentric circles. In order to extend it to a more general configurations, we must use conformal mapping.

Let us consider the second fundamental problem for a circle of radius R. This problem can be reduced to the boundary value problem of the type

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = i\int_0^s (\sigma_{x\nu} + i\sigma_{y\nu})\,ds$$

$$= iR\int_0^\theta (\sigma_{x\nu} + i\sigma_{y\nu})\,d\theta = f_1 + if_2. \tag{4.49}$$

In this case the principal vector and the vector moment of external forces must vanish:

$$\int_0^{2\pi} (\sigma_{x\nu} + i\sigma_{y\nu})d\theta = 0, \tag{4.50}$$

$$\int_0^{2\pi} (f_1 dx + f_2 dy) = \int_0^{2\pi} (-f_1 \sin\theta + f_2 \cos\theta)d\theta = 0. \tag{4.51}$$

We shall assume that the functions $\sigma_{x\nu}$ and $\sigma_{y\nu}$ are continuous functions of the polar angle θ, which allows us to represent the right-hand side of (4.49) in the form of a series

$$f_1 + if_2 = \sum_{-\infty}^{\infty} A_n e^{in\theta}. \tag{4.52}$$

Since the functions $\varphi(z)$ and $\psi(z)$ are analytical inside the circle, they can also be represented in the form of series

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^k, \qquad \psi(z) = \sum_{k=0}^{\infty} a'_k z^k.$$

Next we have

$$\overline{\varphi'(z)} = \sum_{k=1}^{\infty} k\bar{a}_k \bar{z}^{k-1}, \qquad \overline{\psi(z)} = \sum_{k=0}^{\infty} \bar{a}'_k \bar{z}^k.$$

If we assume that the above series converge on the contour as well, we get

$$\sum_{k=1}^{\infty} a_k t^k + t \sum_{k=1}^{\infty} k\bar{a}_k \left(\frac{R^2}{t}\right)^{k-1}$$

$$+ \sum_{k=0}^{\infty} \bar{a}'_k \left(\frac{R^2}{t}\right)^k = \sum_{-\infty}^{\infty} A_k e^{ik\theta}. \tag{4.53}$$

Since the right-hand side of (4.53) is a function of θ, we can go over to this variable also on the left-hand side of this equality:

$$\sum_{k=1}^{\infty} a_k R^k e^{ik\theta} + a_1 R e^{i\theta}$$

$$+ \sum_{k=0}^{\infty} (k+2)\bar{a}_{k+2} R^{k+2} e^{-ik\theta} + \sum_{k=0}^{\infty} \bar{a}'_k R^k e^{-ik\theta} = \sum_{-\infty}^{\infty} A_k e^{ik\theta}.$$

Comparing the coefficients of $e^{i\theta}$, $e^{ik\theta}$ $(k > 1)$ and $e^{-ik\theta}$ $(k \geqslant 0)$, we obtain

$$a_1 + \bar{a}_1 = \frac{A_1}{R}, \tag{4.54}$$

$$a_k R^k = A_k \quad (k > 1), \tag{4.55}$$

$$(k + 2)\bar{a}_{k+2} R^{k+2} + R^k \bar{a}'_k = A_{-k} \quad (k \geqslant 0). \tag{4.56}$$

It follows from (4.54) that $\operatorname{Im} A_1 = 0$. Let us clarify the meaning of this restriction. We have

$$2\pi A_1 = \int_0^{2\pi} (f_1 + if_2) e^{-i\theta} d\theta$$
$$= \int_0^{2\pi} (f_1 \cos\theta + f_2 \sin\theta)\, d\theta + i \int_0^{2\pi} (f_2 \cos\theta - f_1 \sin\theta) d\theta.$$

Considering (4.52), we find that the imaginary part is equal to the magnitude of the moment of external forces and hence is equal to zero. Therefore, Eq. (4.54) is always solvable. True, the value of $\operatorname{Im} a_1$ is not determined. It can be chosen arbitrarily, and this will not affect the state of stress. Further, all values of a_k are found from (4.55) and coefficients a'_k, from (4.56).

Finally, it remains to be seen whether the series obtained in this way actually converge and lead to the solution of the boundary value problem. In view of the simplicity of the proof, it will be carried out under a more stringent condition than the one used while constructing the solution. We shall require that the functions $\sigma_{x\nu}$ and $\sigma_{y\nu}$ have the first derivatives satisfying the Dirichlet conditions (see Sec. 1, Ch. 1, Vol. 1). In this case the series

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^k, \qquad \varphi'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}, \; \psi(z) = \sum_{k=0}^{\infty} a'_k z^k$$

are absolutely convergent. Let us consider the series composed of the moduli of the above series on the circle $|z| = R$:

$$\sum |a_k| R^k, \qquad \sum k\, |a_k|\, R^{k-1}, \qquad \sum |a'_k| R^k.$$

Taking into account the restrictions imposed on the functions $\sigma_{x\nu}$ and $\sigma_{y\nu}$, we find the estimates

$$|A_k, A_{-k}| < \frac{C}{k^3} \quad (k = 1, 2, \dots).$$

With the help of these estimates and considering (4.54), (4.55), and (4.56), we get

$$|a_k| R^k < \frac{C}{k^3}, \quad k|a_k| R^{k-1} < \frac{C'}{k^2}, \quad |a'_k| R^k < \frac{C''}{k^2},$$

where C, C', and C'' are constants. From this follows the convergence of the series.

The solution for the first fundamental problem is also constructed in the same way. The only difference is that a relation similar to (4.54) will be always satisfied.

In certain very important cases, the series method directly leads to a solution in the explicit form. By way of an example, let us consider a plane with a hole of radius

R. We put $\sigma_{x\nu} = P_x/2\pi R$ and $\sigma_{y\nu} = P_y/2\pi R$. We shall directly seek the solution for the boundary conditions in terms of stresses $N_\nu - iT_\nu$ (without going over to the functions f_1 and f_2). In other words, we shall determine the functions $\Phi(z)$ and $\Psi(z)$ rather than $\varphi(z)$ and $\psi(z)$ (the procedure for finding the solution remains the same). It is obvious that P_x and P_y are the projections of the principal vector onto the x- and y-axes. Expressing the boundary conditions in terms of N_ν and T_ν, which are the projections onto the normal and the tangent to the contour, we get

$$N_\nu = -\frac{1}{2\pi R}(P_x \cos\theta + P_y \sin\theta),$$

$$T_\nu = -\frac{1}{2\pi R}(-P_x \sin\theta + P_y \cos\theta)$$

or

$$N_\nu - iT_\nu = -\frac{1}{2\pi R}(P_x - iP_y)e^{i\theta}.$$

Substituting into the boundary conditions the expressions for the functions $\Phi(z)$ and $\Psi(z)$ in the form of series in negative powers of z, we observe that the series are truncated. We shall give here the final result only:

$$\Phi(z) = -\frac{P_x + iP_y}{2\pi(1+\varkappa)}\frac{1}{z},$$

$$\Psi(z) = \frac{\varkappa(P_x - iP_y)}{2\pi(1+\varkappa)}\frac{1}{z} - \frac{P_x + iP_y}{\pi(1+\varkappa)}\frac{R^2}{z^3}. \tag{4.57}$$

Assuming that the radius $R \to 0$, and the resultant of (P_x, P_y) remains unchanged, we get

$$\Phi(x) = -\frac{P_x + iP_y}{2\pi(1+\varkappa)}\frac{1}{z}, \quad \Psi(z) = \frac{\varkappa(P_x - iP_y)}{2\pi(1+\varkappa)}\frac{1}{z}. \tag{4.58}$$

This solution should be treated as the solution for a concentrated force (P_x, P_y) applied at the origin. The corresponding expressions for the functions $\varphi(z)$ and $\psi(z)$ are given below:

$$\varphi(z) = -\frac{P_x + iP_y}{2\pi(1+\varkappa)}\ln z, \quad \psi(z) = \frac{\varkappa(P_x - iP_y)}{2\pi(1+\varkappa)}\ln z. \tag{4.59}$$

If the concentrated force is applied at a point other than the origin, we use the transformation formulas for the translation of coordinates. This leads to the expressions for $\varphi(z)$ and $\psi(z)$, when the concentrated force is applied at an arbitrary point z_0:

$$\varphi(z) = -\frac{P_x + iP_y}{2\pi(1+\varkappa)} \ln(z - z_0),$$
$$\psi(z) = \frac{\varkappa(P_x - iP_y)}{2\pi(1+\varkappa)} \ln(z - z_0) + \frac{\bar{z}_0(P_x + iP_y)}{2\pi(1+\varkappa)} \frac{1}{z - z_0}. \tag{4.60}$$

Section 5
The Method of Conjugation

The above discussion shows that the contact problems (as well as problems in the theory of elasticity for a body with a cut, see Sec. 6) may be reduced to singular integral equations whose solution, in turn, may be reduced to Riemann's boundary value problem. However, in some special cases, the problem can be directly reduced to Riemann's boundary value problem [17].

We shall now describe the method called the method of conjugation. Let D^+ and D^- be an upper and a lower half-plane. We consider a certain function $\Phi(z)$ which is analytical in the domain D^+. We introduce a function in the domain D^-, denoted by $\bar{\Phi}(z)$ and defined by the formula

$$\bar{\Phi}(z) = \varphi(\bar{z}) \quad (z \in D^+, \bar{z} \in D^-). \tag{5.1}$$

Thus, the continuation of a function defined above will be called a continuation by conjugation. Similarly, an analytical function in D^- may be continued in D^+. It can be shown that the function $\bar{\Phi}(z)$ is analytical in D^-. Obviously, the following relation holds between the limiting values (on the real axis) of the functions $\Phi(z)$ and $\bar{\Phi}(z)$:

$$\bar{\Phi}^-(t) = \overline{\Phi^+(t)}, \qquad \bar{\Phi}^+(t) = \overline{\Phi^-(t)}. \tag{5.2}$$

It follows from this relation that if the condition $\operatorname{Im}\Phi^+(t) = 0$ is satisfied, on some part of the real axis, the function $\bar{\Phi}(z)$ will be analytical continuation of the function $\Phi(z)$ in the domain D^-, and vice versa.

Let us now consider a problem in the theory of elasticity for the half-plane D^- under the conditions at infinity introduced earlier:

$$\Phi(z) = -\frac{P_x + iP_y}{2\pi} \frac{1}{z} + O\left(\frac{1}{z^2}\right),$$
$$\Psi(z) = \frac{P_x - iP_y}{2\pi} \frac{1}{z} + O\left(\frac{1}{z^2}\right), \tag{4.17}$$

where P_x and P_y are the projections of the principal vector of external forces onto the x- and y-axes.

In the same way as in Eq. (5.1), we continue the functions $\Phi(z)$ and $\Psi(z)$ in the domain D^+ by conjugation. We denote the functions thus obtained by $\bar{\Phi}(z)$ and $\bar{\Psi}(z)$, and introduce a new analytical function $\Phi_1(z)$ in the same domain D^+:

$$\Phi_1(z) = -\bar{\Phi}(z) - z\bar{\Phi}'(z) - \bar{\Psi}(z) \qquad (z \in D^+). \tag{5.3}$$

Continuing both sides of Eq. (5.3), we get the relation

$$\bar{\Phi}_1(z) = -\Phi(z) - z\Phi'(z) - \Psi(z) \tag{5.4}$$

or

$$\Psi(z) = -\bar{\Phi}_1(z) - \Phi(z) - z\Phi'(z). \tag{5.4'}$$

With the help of the formula (5.4′), we can go over to the functions $\Phi(z)$ and $\Phi_1(z)$ in the representation (2.8) for stresses. The first of these functions is analytical in D^-, while the second is analytical in D^+. The corresponding expression for tne combination of stresses $\sigma_y - i\tau_{xy}$, which we shall be needing in future, can be written as follows:

$$\sigma_y - i\tau_{xy} = \Phi(z) + (z - \bar{z})\overline{\Phi'(z)} - \Phi_1(\bar{z}). \tag{5.5}$$

Further, we assume that the function $\Phi(z)$ has limiting values $\Phi^-(t)$ at nearly all points on the real axis, with the exception of just a few, at which the equality $\lim_{z \to t} y\overline{\Phi'(z)} = 0$ is nevertheless satisfied. Then, carrying out the limit transition to the points on the real axis in Eq. (5.5) and neglecting the subscript on the function[8] $\Phi_1(z)$, we arrive at Riemann's boundary value problem

$$\Phi^-(t) - \Phi^+(t) = (\sigma_y - i\tau_{xy})|_{\operatorname{Im} z=0} = f(t), \tag{5.6}$$

where $f(t)$ is a given function.

Thus, a solution of the second fundamental problem in the theory of elasticity can be found directly for a half-plane. Indeed, the piecewise analytical function $\Phi(z)$ can be reoresented in the form

$$\Phi(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)\,dt}{t - z}. \tag{5.7}$$

Naturally, in order to determine stresses at any point z, we must find the function $\Phi(z)$ at the conjugate point as well.

In order to go over to a consideration of the first fundamental problem, we must obtain a relation between the functions $\Phi(z)$ and $\Psi(z)$ at the boundary, and not between $\varphi(z)$ and $\psi(z)$, as in (2.7). Hence, differentiating (2.7) with respect to the variable x, we obtain (the derivatives of displacements with respect to x are denoted by u' and v')

$$\varkappa\Phi(z) - \overline{\Phi(z)} - z\overline{\Phi'(z)} - \overline{\Psi(z)} = 2\mu(u' + iv'). \tag{5.8}$$

Eliminating $\Psi(z)$ in accordance with (5.4′), we get the relation

$$\varkappa\Phi(z) + (\bar{z} - z)\overline{\Phi'(z)} + \Phi_1(z) = 2\mu(u' + iv'). \tag{5.9}$$

Performing the limit transition, we get Riemann's boundary value problem

$$\varkappa\Phi^-(t) + \Phi^+(t) = 2\mu(u' + iv')|_{\operatorname{Im} z=0} = f(t). \tag{5.10}$$

---

[8] Henceforth, the functions $\Phi(z)$ and $\Phi_1(z)$ will be considered as the values of the same piecewise analytical function $\Phi(z)$ in D^- and D^+ respectively.

In order to solve this equation, we must resort to an auxiliary piecewise analytical function, which we denote by $\Omega(z)$ and define as follows:

$$\Omega(z) = \Phi(z) \qquad (y > 0), \qquad \Omega(z) = -\varkappa\Phi(z) \quad (y < 0). \tag{5.11}$$

We then get the following boundary value problem for this function:

$$\Omega^+(t) - \Omega^-(t) = f(t). \tag{5.12}$$

The solution of this problem can be easily obtained:

$$\Omega(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)\,dt}{t - z}. \tag{5.13}$$

Of course, the application of the conjugation method for solving contact problems for a half-plane is of considerable interest. Suppose that we are given a set of points $a_1, b_1, a_2, b_2, \dots, a_n, b_n$ on the real axis (the boundary of a half-plane). The displacements are given on the sections a_j, b_j (system of arcs L) (in other words, we consider the problem of indentation of punches with a coupling), while the stresses are given on the intervals $(-\infty, a_1)$, (b_j, a_{j+1}), (b_n, ∞) (system of arcs M).

We shall assume that the stresses vanish in the system of arcs M. These may be eliminated with the help of a particular solution of the second fundamental problem by specifying on the arcs L, for example, additional stresses equal to zero. As a result of superposition, we get the required homogeneous boundary conditions on the system M, while the corresponding changes in the boundary conditions will take place on the system L. In accordance with (5.6) and (5.11), we obtain Riemann's boundary value problem with discontinuous coefficients

$$\begin{aligned} \Phi^-(t) - \Phi^+(t) &= 0 \qquad (t \in M), \\ \varkappa\Phi^-(t) + \Phi^+(t) &= f(t) \qquad (t \in L). \end{aligned} \tag{5.14}$$

Since the right-hand sides in the system of arcs L are defined by the derivatives of the given displacements, the boundary value problem is actually formulated for the case when the displacements of punches are given to within translational shift. This shift is determined during the solution of the entire problem from the additional conditions concerning the applied forces. The magnitude of the moments is determined after finding the indentation distribution.

The solution of the problem (5.14) has the form

$$\begin{gathered} \Phi(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(t)\,dt}{X(t)(t-z)} + \frac{1}{2}X(z)P_{n-1}(z), \\ X(z) = \prod_{k=1}^{n}(z - a_k)^{-\frac{1}{2}+i\beta}(z - b_k)^{\frac{1}{2}-i\beta} \\ \left(\beta = \frac{1}{2\pi}\ln\varkappa\right). \end{gathered} \tag{5.15}$$

Here, $P_{n-1}(z)$ is a polynomial of a degree not exceeding $n - 1$, containing n complex constants c_j, and $X(t)$ are the values assumed by the function $X(z)$ on the left-hand side. This function is single-valued in a plane with a system of cuts L, and satisfies the condition $z^n X(z) \to 1$ as $z \to \infty$.

According to the Sochozky-Plemelj formula, we obtain, for the system of cuts L,

$$\Phi^+(t_0) = \frac{1}{2} f(t_0) + \frac{X(t_0)}{2\pi i} \int_L \frac{f(t)dt}{X(t)(t - t_0)} + \frac{X(t_0)}{2} P_{n-1}(t_0). \qquad (5.15')$$

To determine the contact stresses (in accordance with (5.6)), it is necessary to construct the function $\Phi^-(t)$. The simplest way of doing this is to start the second condition in (5.14). This finally gives

$$P + iT = \frac{\varkappa - 1}{2\varkappa} f(t_0) + \frac{\varkappa + 1}{\varkappa} \frac{X(t_0)}{2\pi i}$$

$$\times \int_L \frac{f(t)dt}{X(t)(t - t_0)} + \frac{\varkappa + 1}{2\varkappa} X(t_0) P_{n-1}(t_0) \qquad (t \in L). \qquad (5.15'')$$

The solution (5.15) is the most general solution when unlimited (but of course integrable) singularities are assumed at all end points. This solution contains arbitrary constants c_j.

We shall now determine these constants. For a set of punches, we shall consider two formulations of the problems. In the first case the displacement of punches takes place independently, and the magnitudes of the principal vectors of forces applied to each punch serve as additional conditions. In the second case, however, only the total principal vector of forces is known.

For the first case, we obtain the equalities

$$\int_{(a_j, b_j)} [P(t_0) + iT(t_0)]dt_0 = -P_{xk} + iP_{yk} \qquad (k = 1, 2, \dots, n). \qquad (5.16)$$

This is a system of n complex equations in n complex unknowns.

In the second case, we proceed from the equalities

$$0 = [u(a_{k+1}) + iv(a_{k+1})] - [u(b_k) + iv(b_k)]$$

$$= \int_{b_k}^{a_{k+1}} (u'^- + iv'^-)dt_0 \qquad (k = 1, 2, \dots, n - 1), \qquad (5.17)$$

which indicate that the distance between the end points of punches remains constant under indentation. In order to construct these equalities, we shall use the expressions for the derivatives of displacements on the arcs of the system M:

$$2\mu(u'^- + iv'^-)$$

$$= (\varkappa + 1)\Phi(t_0) = (\varkappa + 1)\Phi_0(t_0) + (\varkappa + 1)X(t_0)P_{n-1}(t_0). \quad (5.18)$$

The system (5.17) contains only $n - 1$ equations. Yet another equation is obtained from the conditions imposed on the principal vector of forces $P_x + iP_y$ applied to the set of punches. This condition may be directly expressed in terms of the function $\Phi(z)$:

$$\lim_{z \to \infty} z\Phi(z) = -\frac{1}{2\pi}(P_x + iP_y).$$

This gives

$$c_{n-1} = -\frac{1}{2\pi}(P_x + iP_y). \tag{5.19}$$

The above results can be used for considering the case when the stresses at the end points are assumed to be limited. The condition of boundedness of stresses is equivalent to the condition that they be equal to zero (if $f(t)$ belongs to the class H-L). The expression for the contact pressure may be obtained in a form similar to the formula (5.15″) or, alternatively, by proceeding directly from the formula (5.15), but by introducing additional restrictions on the expression for $X(z)$. The problem is thus reformulated, and in order that it be solvable, we must assume[9] that the position of the points a_j and b_j is also unknown.

Let us now consider the problem of the indentation of a system of smooth punches (in the absence of friction). In this case also, we assume that the first condition in (5.14) is satisfied for the system of arcs M. On the other hand, we assume that the shearing stresses in the system of arcs L are equal to zero and that the normal displacements are known (possibly, to within the real constants). In other words,

$$v(t) = g(t) + c_k \quad (t \in (a_k, b_k)). \tag{5.20}$$

Thus, τ_{xy} is equal to zero on the entire real axis.

In this case also, two formulations of the problem are assumed. In the first case, the punches in the system are rigidly connected and the force P_y, applied to the entire system, is given. In the second case, the displacement of each punch is assumed to be independent and the forces P_{yk} applied to each punch are specified.

We shall use the condition that the stress τ_{xy} is equal to zero on the entire axis $\mathrm{Im}\ z = 0$. Combining the condition (5.6) with the one obtained from it by a transition to the conjugate (taking (5.2) into account), we arrive at the expression

$$-2i\tau_{xy} = \Phi^-(t) + \bar{\Phi}^-(t) - \Phi^+(t) - \bar{\Phi}^+(t) = 0. \tag{5.21}$$

Consequently, we get

$$\Phi^+(t) + \bar{\Phi}^+(t) = \Phi^-(t) + \bar{\Phi}^-(t), \tag{5.22}$$

from which it follows that the function $\Phi(z) + \bar{\Phi}(z)$ is analytical on the entire plane. Together with the condition (4.17), this means that this function is equal to zero. Thus we have

$$\bar{\Phi}(z) = -\Phi(z). \tag{5.23}$$

[9] This is in accordance with the physical formulation of this kind of contact problems.

As before, the boundary conditions for the derivatives of the displacements along the x-axis can be written in the form

$$2\mu v'(t) = f_1(t) \qquad (t \in L). \tag{5.24}$$

With the help of the expression on the left-hand side of (5.10), the condition (5.24), as applied to the function $\Phi(z)$, can be written in the form

$$\frac{4\mu i v'}{\varkappa + 1} = \Phi^-(t) + \Phi^+(t) = \frac{2f_1(t)}{\varkappa + 1} = f(t) \quad (t \in L). \tag{5.25}$$

Similarly, we can rewrite the conditions (5.6):

$$\Phi^-(t) - \Phi^+(t) = 0 \qquad (t \in M).$$

In this way, we have obtained Riemann's boundary value problem, whose solution is the function

$$\Phi(z) = \frac{1}{\pi X(z)} \int_L \frac{X(t)f(t)dt}{t - z} + \frac{P_{n-1}(z)}{X(z)}, \tag{5.26}$$

$$X(z) = \sqrt{(z - a_1)(z - b_1) \dots (z - a_n)(z - b_n)},$$

where by $X(z)$ we mean the branch which is single-valued on the plane cut along L and such that $z^{-n}X(z) \to 1$ as $|z| \to \infty$. Moreover,

$$X(t) = \sqrt{(t - a_1)(t - b_1) \dots (t - a_n)(t - b_n)}.$$

We shall show that the first term satisfies the remaining condition (5.23). We have

$$\bar{\Phi}(z) = \frac{1}{\pi \bar{X}(z)} \int_L \frac{\overline{X(t)}f(t)dt}{t - z}.$$

It follows from the structure of the function $X(z)$ that $\bar{X}(z) = \pm X(z)$. In order to find the sign, we must consider the behaviour of these functions for large values of $|z|$. Since $\bar{X}(z)$ and $X(z)$ are of the order of z^n at infinity, it follows that the upper sign (plus) must be retained. Since the points in the system of arcs L lie on the real axis, $\bar{X}(t) = -X(t)$. Thus, our statement has been proved. The second term will satisfy this condition if all the coefficients of the polynomial are imaginary numbers. For the pressure created by the punch, we obtain the formula

$$P(t_0) = \frac{2}{\pi X(t_0)} \int_L \frac{X(t)f(t)dt}{t - t_0} + \frac{2P_{n-1}(t_0)}{X(t_0)}. \tag{5.27}$$

While determining the coefficients of polynomial with apparent variations, we use the same considerations as were used in the case of punches with coupling. For example, when the system of punches forms a single entity, only the difference between the vertical displacements of the points b_j and a_{j+1} should be put equal to zero.

It should be observed that the physical formulation of the problem of indentation by a smooth punch requires an additional restriction on the solution in that the contact stress must be negative. Otherwise, a gap is created between the punch and an elastic body, and this considerably changes the formulation of the boundary value problem—the point at which the punch ceases to be in contact with the body becomes mobile and will be determined in the course of the solution of the problem (see [17] for details).

We now consider the contact problem for two elastic half-planes having different parameters. This case may serve as the basis for considering the contact problem for two bodies of quite arbitrary configuration, if the area of contact is small in comparison with the dimensions of the bodies. In this case we should independently use the solutions for each half-plane, and formulate Riemann's boundary value problem from the condition that on the boundary the contact stresses and displacements are equal. Consequently, as in the general three-dimensional case, we actually arrive at the problem of a rigid punch in a half-plane, when the profile of the punch depends in a definite way on the profile of each of the elastic bodies and their elastic constants.

Let us now consider a problem in the theory of elasticity for a plane with cuts, using the method of conjugation. We shall first formulate the boundary value problem. Suppose that we are given a set of cuts $L_1, L_2, \dots, L_m$, lying on the same straight line which is taken as the real axis. On the sides of the cuts are given the loads

$$\sigma_y^+, \tau_{xy}^+, \sigma_y^-, \tau_{xy}^-. \tag{5.28}$$

The plus sign corresponds to the upper edges of the cuts, while the minus sign corresponds to the lower ones. As before, we start with the representation (4.17). Moreover, the quantities P_x and P_y have the same meaning as before, i.e. they are the projections of the principal vector of forces applied to the sides of the cuts. The function $\Phi(z)$, as well as the function $\Psi(z)$, is now analytical over the entire plane with the exception of the cuts. We shall follow a procedure which, by analogy, is also called the method of continuation by conjugation, but which now consists in the construction of the functions $\bar{\Phi}(z) = \overline{\Phi(\bar{z})}$ and $\bar{\Psi}(z) = \overline{\Psi(\bar{z})}$ in the entire plane. We form a new function

$$\Omega(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z). \tag{5.29}$$

Eliminating the function $\Psi(z)$ in (5.5) with the help of (5.29), we get

$$\Phi(z) + (z - \bar{z})\bar{\Phi}'(z) + \Omega(\bar{z}) = \sigma_y - i\tau_{xy}. \tag{5.30}$$

We now perform the limit transition (under the same restrictions on the behaviour of the function at the end points as before), and arrive at the following system of Riemann's boundary value problems for the functions $\Phi(z)$ and $\Omega(z)$:

$$\begin{aligned} \Phi^+(t) + \Omega^-(t) &= \sigma_y^+ - i\tau_{xy}^+ = p(t), \\ \Phi^-(t) + \Omega^+(t) &= \sigma_y^- - i\tau_{xy}^- = q(t). \end{aligned} \tag{5.31}$$

This system can be transformed to give a set of two Riemann's boundary value problems:

$$
\begin{aligned}
&[\Phi + \Omega]^{+} + [\Phi + \Omega]^{-} = p + q, \\
&[\Phi - \Omega]^{+} - [\Phi - \Omega]^{-} = p - q.
\end{aligned} \tag{5.32}
$$

The solution of each of these problems has been obtained above.

The constants appearing in the general solution are determined from the condition that the displacements are single-valued for each round of the cuts.

It should be observed that the method of conjugation can be effectively used for solving contact problems when the elastic body is bounded by the arc of a circle, and for problems when the cuts are situated along the arc of one circle. In this case, the function $\overline{\Phi(1/z)}$ serves as the continuation of the function by conjugation.

Before concluding the section, let us consider some specific examples illustrating the method of conjugation.

Let us consider the indentation problem for a punch of length $2l$ with a straight line as the foundation ($f(t) = 0$) under the conditions $P_x = 0$ and $P_y = -P_0$. In this case we immediately obtain a simple solution with the help of the formulas (5.15) and (5.15′):

$$
\Phi(z) = \frac{iP_0}{2\pi}(z + l)^{-\frac{1}{2} + i\beta}(z - l)^{-\frac{1}{2} - i\beta}. \tag{5.33}
$$

The final expressions for the contact pressures are

$$
P(t) = \frac{P_0(1 + x)}{\pi\sqrt{l^2 - t^2}\sqrt{x}} \cos\left[\frac{\ln x}{2\pi}\ln\frac{l + t}{l - t}\right], \tag{5.34}
$$

$$
T(t) = \frac{P_0(1 + x)}{\pi\sqrt{l^2 - t^2}\sqrt{x}} \sin\left[\frac{\ln x}{2\pi}\ln\frac{l + t}{l - t}\right]. \tag{5.35}
$$

It should be noted that, as expected, the asymptotics of the expressions for the contact pressures are identical to the one obtained by using the solution of the transcendental equation (9.37), Ch. 3, Vol. 1, for a wedge of angle π when displacements are given on one side, and stresses on the other.

It should be remarked that as we approach the end points, the contact stresses change sign an infinite number of times, but the oscillating zone is extremely small and constitutes just 0.0003 of the length of the punch.

Let us now consider in greater detail the indentation problem for a smooth punch (Fig. 35) (we omit the subscripts for the points a and b). In accordance with (5.27), the pressure can be represented in the form

$$
P(t_0) = \frac{2}{\pi\sqrt{(t_0 - a)(b - t_0)}} \int_a^b \frac{\sqrt{(t - a)(b - t)}}{t - t_0} f(t)dt + \frac{2D}{\sqrt{(t_0 - a)(b - t_0)}}, \tag{5.36}
$$

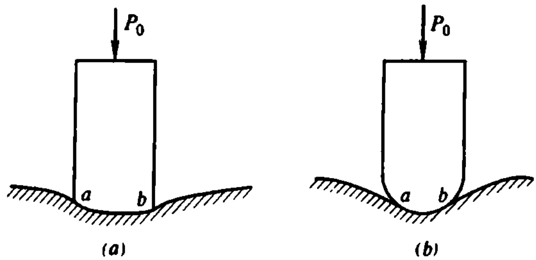

Fig. 35. Indentation of a punch into a half-plane: (a) complete contact, (b) partial contact.

where the real constant D may be obtained from the condition (5.19): $D = P/2\pi$. After the solution has been formally constructed, it is necessary to verify that the condition $P \leqslant 0$ is satisfied.

Suppose that the contact between the punch and the half-plane is not complete. Then the position of the points a and b becomes unknown. However, even in this case, the expression (5.36) can be used to obtain a comprehensive result. The fact that the position of the boundary points of contact are unknown implies that the following conditions must be satisfied:

$$P(a) = P(b) = 0. \tag{5.37}$$

We shall show that these conditions are equivalent to the conditions of boundedness of the solution. The boundedness of a solution implies that it vanishes. Introducing the notation $Q(t) = (t - a)(b - t)$, we can rewrite (5.36) in the form

$$\begin{aligned}
&\frac{2}{\pi\sqrt{Q(t_0)}}\int_a^b \frac{[Q(t) - Q(t_0)]f(t)\,dt}{\sqrt{Q(t)}(t - t_0)} \\
&\qquad + \frac{2\sqrt{Q(t_0)}}{\pi}\int_a^b \frac{f(t)\,dt}{\sqrt{Q(t)}(t - t_0)} + \frac{2D}{\sqrt{Q(t_0)}} \\
&\qquad = \frac{2\sqrt{Q(t_0)}}{\pi}\int_a^b \frac{f(t)\,dt}{\sqrt{Q(t)}(t - t_0)} + \frac{At_0 + B + 2D}{\sqrt{Q(t_0)}},
\end{aligned} \tag{5.38}$$

$$A = -\frac{2}{\pi}\int_a^b \frac{f(t)\,dt}{\sqrt{Q(t)}},$$

$$B = -\frac{2}{\pi}\int_a^b \frac{tf(t)\,dt}{\sqrt{Q(t)}} + \frac{2(a + b)}{\pi}\int_a^b \frac{f(t)\,dt}{\sqrt{Q(t)}}.$$

Using the estimates (1.25), Ch. 1, Vol. 1, we can show that the first term in (5.38) vanishes at the points a and b. Hence the condition of boundedness at these points should be verified only for the last term. It is necessary to require that $A = B + 2D = 0$, which means that the entire term is identically equal to zero. Finally, we get

$$P(t_0) = \frac{2}{\pi}\sqrt{(t-a)(b-t)}\int_a^b \frac{f(t)\,dt}{(t-t_0)\sqrt{(t-a)(b-t)}}. \qquad (5.39)$$

We shall now give the solution for the case when the contour of the punch is a parabola $v(t) = kt^2$. Due to symmetry, the points a and b will be situated at the same distance l from the y-axis. Suppose that the entire arc$(-l, l)$ is initially in contact with the half-plane. This gives

$$\Phi(z) = \frac{4k\mu}{\pi(\varkappa+1)}\,\frac{1}{\sqrt{l^2-z^2}}\int_{-l}^{l}\frac{t\sqrt{l^2-t^2}}{t-z}dt + \frac{P_0}{2\pi\sqrt{l^2-z^2}}. \qquad (5.40)$$

In order to evaluate the integral, we use the formula[10] (1.61), Ch. 1, Vol. 1:

$$\int_{-l}^{l}\frac{t\sqrt{l^2-t^2}}{t-z}dt = \pi i\left(z\sqrt{l^2-z^2} + iz^2 - \frac{il^2}{2}\right).$$

The expression for the stress then has the form

$$P(t) = \frac{l^2 - 2t^2}{\sqrt{l^2-t^2}} + \frac{P_0}{\pi\sqrt{l^2-t^2}}. \qquad (5.41)$$

The condition $P(t) \geqslant 0$ $(-l \leqslant t \leqslant l)$ leads to the following restrictions: $P_0 \geqslant 4\pi\mu k l^2/(\varkappa + 1)$. Otherwise, the entire surface of the punch will not come in contact with the elastic half-plane. Denoting the end points of the arc of contact by a' and b' $(-a' = b' = l')$, replacing l by l' in (5.41), and requiring that $P(a') = P(b') = 0$, we can immediately write the result

$$l' = \frac{\sqrt{P_0(\varkappa+1)}}{\sqrt{4\pi\mu k}}. \qquad (5.42)$$

Taking this into account, we get the following expression for pressure from Eq. (5.41):

$$P(t) = \frac{8\mu k}{\varkappa+1}\sqrt{l'^2 - t^2}. \qquad (5.43)$$

---

[10] Here we have used the asymptotic representations for large values of $|z|$: $z\sqrt{l^2-z^2} = -iz^2 + \frac{il^2}{2} + O\left(\frac{1}{z}\right)$.

Section 6
Problems for Bodies with Cuts (General Case)

Let us assume that an elastic body has a set of cuts whose borders are the open contours L_j ($j = 1, 2, \ldots, m$). We denote the ends of these contours by a_k and b_k, and choose the direction from a_k to b_k as positive. Consequently, the left- and the right-hand sides of a cut are given the signs "+" and "−" respectively. We assume that the stresses are given on the sides of the cuts[11]. Further, it is assumed that the principal vector of forces applied to each cut vanishes (if this is not the case, a suitable transformation of the boundary conditions should be carried out; see Sec. 2). While formulating the boundary conditions, we shall first use the integrals (2.9), and introduce the notation $f^+(t^+)$ and $f^-(t^-)$ for the boundary conditions in order to distinguish between points having the same coordinates, but lying on different sides of a cut.

Thus, the problem of the theory of elasticity for a body with cuts (for the sake of simplicity we shall consider an unbounded body) is reduced to determining, in the plane of the complex variable z, the functions $\varphi(z)$ and $\psi(z)$ which are analytical everywhere except at the cuts and satisfy the conditions

$$\left.\begin{aligned} \varphi^+(t^+) + t\overline{\varphi^{+\prime}(t^+)} + \overline{\psi^+(t^+)} &= f^+(t^+) + c_j \qquad (6.1)\\ \varphi^-(t^-) + t\overline{\varphi^{-\prime}(t^-)} + \overline{\psi^-(t^-)} &= f^-(t^-) + c_j \qquad (6.2) \end{aligned}\right\} \quad (t \in L_j).$$

The constants c_j are not known beforehand (one of them may be assumed to be given like in the case for multiply connected domains) and are determined in the course of the solution from the condition that the displacements are single-valued. First, we simplify the formulation of the problem by assuming that the functions $f^+(t^+)$ and $f^-(t^-)$ coincide. For this purpose, we introduce the following functions in the whole plane:

$$\varphi_0(z) = 0, \qquad \psi_0(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{L_j} \frac{f^-(t) - f^+(t)}{t - z} dt$$

and go over to new functions $\varphi^*(z) = \varphi(z) - \varphi_0(z)$ and $\psi^*(z) = \psi(z) - \psi_0(z)$. The boundary conditions for these functions can then be expressed in the following form (we shall be omitting the index "*"):

$$\varphi^+(t) + t\overline{\varphi^{+\prime}(t)} + \overline{\psi^+(t)} = \varphi^-(t) + t\overline{\varphi^{-\prime}(t)} + \overline{\psi^-(t)} = f(t) + c_j \quad (6.3)$$

$$\left(\frac{1}{2}\,[f^+(t) + f^-(t)] = f(t)\right).$$

With the help of an auxiliary function $\omega(t)$ we shall now derive a singular equation corresponding to the problem formulated above.

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[11] From the point of view of applications, this is the most interesting case.

On the contours L_j, we introduce the auxiliary functions $\omega_j(t)$, which are defined as follows:

$$\omega_j(t) = \varkappa\varphi^+(t) - t\overline{\varphi^{+\prime}(t)} - \overline{\psi^+(t)}$$
$$- \varkappa\varphi^-(t) + t\overline{\varphi^{-\prime}(t)} + \overline{\psi^-(t)} \qquad (t \in L_j). \tag{6.4}$$

These functions represent the jump in the displacement vector (to within the factor 2μ); hence they must vanish at the ends of the segments.

By direct substitution, and using the Sochozky-Plemelj formula (1.14), Ch. 1, Vol. 1, we can verify that the functions

$$\varphi(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \frac{1}{1+\varkappa} \int_{L_j} \frac{\omega_j(t)}{t-z} dt, \tag{6.5}$$

$$\psi(z) = -\sum_{j=1}^{m} \frac{1}{2\pi i} \frac{1}{1+\varkappa} \int_{L_j} \frac{\overline{\omega_j(t)} + \bar{t}\omega_j'(t)}{t-z} dt \tag{6.6}$$

identically satisfy the first group of equalities (6.3), expressing the condition of continuity of the stress vector. The second group of conditions, determined by the very stresses at the cuts, leads to the following system of singular integral equations:

$$\sum_{j=1}^{m} \left\{ \frac{1}{2\pi i} \int_{L_j} \frac{\omega_j(t)}{t-t_0} dt + \frac{1}{2\pi i} \int_{L_j} \omega_j(t) d \ln \frac{\bar{t}-\bar{t}_0}{t-t_0} \right.$$
$$\left. - \frac{1}{2\pi i} \int_{L_j} \overline{\omega_j(t)} d \frac{\bar{t}-\bar{t}_0}{t-t_0} \right\} = (1+\varkappa)[f_j(t_0) + c_j] \quad (t_0 \in L_j). \tag{6.7}$$

While deriving these equations, we have used integration by parts and the condition $\omega_j(a_j) = \omega_j(b_j) = 0$. Since only such solutions of Eq. (6.7) are sought that are bounded at the ends (it can be shown that in view of the boundedness, the solution vanishes), the index of the equation will be equal to $-m$. This means that the equation can be solved only if the condition (3.14), Ch. 1, Vol. 1 is satisfied. According to this condition, the right-hand side is orthogonal to all the eigenfunctions of the companion equation (at the ends, these functions become infinite), and this leads to the conditions imposed on the constants c_j.

We obtain a system of $2m$ real equations for complex constants c_j. The solvability of this system of algebraic equations and hence of the integral equation itself in the class of functions under consideration follows from the uniqueness of the solution of the problem in the theory of elasticity. Consequently, the problem in the theory of elasticity also turns out to be solvable.

It should be noted that if the cuts are situated on the same straight line (which is naturally taken as the real axis), Eq. (6.7) is considerably simplified and assumes the form

$$\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{L_j} \frac{\omega_j(t)}{t - t_0} dt = (1 + \varkappa) f_j(t_0) + c_j. \tag{6.8}$$

A somewhat modified equation,obtained by differentiating Eq. (6.8) [31], is often used for obtaining a solution. In this case, the right-hand side contains the stresses themselves:

$$\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{L_j} \frac{\omega'_j(t)}{t - t_0} dt = (1 + \varkappa) f'(t_0) \quad (t_0 \in L_j). \tag{6.8'}$$

Significantly, the required function will now have a singularity at the ends. Hence the index of the equation is equal to m and the solution will contain constants which must be determined from the condition that the displacements are single-valued. It follows from the structure of the characteristic function, as well as from the expression for the eigenfunctions (8.26), Ch. 3, Vol. 1 for an angle 2π, that the singularity is of the order 1/2.

There are no fundamental difficulties in solving problems for bodies with cuts, when the domain under consideration does not occupy the entire plane. Suppose that the body which we are considering is bounded from inside or outside by a certain contour L. We introduce auxilliary functions $\omega_j(t)$ on the cuts (it is assumed that a cut does not appear on the contour) in the same way as before. With the help of these functions, we obtain the following new functions:

$$\varphi_0(z) = \varphi(z) - \sum_{j=1}^{m} \frac{1}{2\pi i(1 + \varkappa)} \int_{L_j} \frac{\omega_j(t)}{t - z} dt,$$

$$\psi_0(z) = \psi(z) + \sum_{j=1}^{m} \frac{1}{2\pi i(1 + \varkappa)} \int_{L_j} \frac{\overline{\omega_j(t)} + \bar{t}\omega'_j(t)}{t - z} dt.$$

It can be shown that the function $\varphi_0(z)$ and $\psi_0(z)$ will be analytical in the entire continuous domain which we shall denote by D. Then, solving somehow the problem obtained above (for conditionally given functions ω_j), we arrive at the representations

$$\varphi_0(z) = H_1(\omega_j, z), \qquad \psi_0(z) = H_2(\omega_j, z).$$

Turning once again to the conditions on the cuts, we obtain equations which differ from Eqs. (6.6) or (6.7) only in their regular terms.

Let us consider another method of constructing integral equations for bodies with cuts. The general case of curvilinear cuts has been discussed in [32]. For the sake of simplicity, we shall confine ourselves to rectilinear cuts only [33].

Suppose that an elastic plane has a cut $|x| < l$, $y > 0$. We assume that there is no stress at infinity. The boundary conditions can be written in the form

$$\sigma_y^{\pm}(x,0) - i\tau_{xy}^{\pm}(x,0) = p(x) \pm q(x), \qquad |x| < l, \tag{6.9}$$

where

$$p(x) = \frac{1}{2}(\sigma_y^+ + \sigma_y^-) - \frac{i}{2}(\tau_{xy}^+ + \tau_{xy}^-), \qquad q(x) = \frac{1}{2}(\sigma_y^+ - \sigma_y^-) - \frac{i}{2}(\tau_{xy}^+ - \tau_{xy}^-).$$

Let us first consider an auxiliary problem. Suppose that the jump in the stress vector and the derivatives (with respect to x) of the jump in the displacement vector are given for the interval $|x| < l$ in a plane; moreover, it is assumed that the jump in displacements vanishes at the end points

$$\sigma_y^+ - \sigma_y^- - i(\tau_{xy}^+ - \tau_{xy}^-) = 2q(x), \tag{6.10}$$

$$\frac{\partial}{\partial x}[u^+ - u^- + i(v^+ - v^-)] = \frac{i(\varkappa + 1)}{2\mu} g'(x). \tag{6.11}$$

We shall solve this problem by the method of conjugates in the same way as the problem for a cut, when the stresses at its ends are known (see Sec. 5). In this case we introduce the function $\Omega(z)$ in place of $\Psi(z)$ according to the formula (5.29). Using the formula (5.8), we can then obtain the following boundary conditions for the functions $\Phi(z)$ and $\Omega(z)$:

$$\begin{gathered} \Phi^+(x) - \Omega^-(x) - \Phi^-(x) - \Omega^+(x) = 2q(x), \\ \varkappa\Phi^+(x) - \Omega^-(x) - \varkappa\Phi^-(x) + \Omega^+(x) = i(\varkappa + 1)g'(x). \end{gathered} \tag{6.12}$$

Adding up (6.12), we immediately arrive at Riemann's problem for the function $\Phi(z)$:

$$\Phi^+(x) - \Phi^-(x) = i\left[g'(x) - i\frac{2q(x)}{\varkappa + 1}\right] = iQ(x). \tag{6.13}$$

Subtracting the second equation from the first (having multiplied it by $\varkappa$), we obtain the following problem for the function $\Omega(z)$:

$$\Omega^+(x) - \Omega^-(x) = i[Q(x) - 2iq(x)]. \tag{6.14}$$

This gives the following representation for each of the functions:

$$\Phi(z) = \frac{1}{2\pi}\int_{-l}^{l} \frac{Q(t)}{t - z}dt, \tag{6.15}$$

$$\Omega(z) = \frac{1}{2\pi}\int_{-l}^{l} \frac{Q(t) - 2iq(t)}{t - z}dt. \tag{6.16}$$

The expression for $\Psi(z)$ has the form

$$\Psi(z) = \frac{1}{2\pi}\int_{-l}^{l}\left[\frac{\overline{Q(t)} - 2i\overline{q(t)}}{t - z} - \frac{tQ(t)}{(t - z)^2}\right]dt. \tag{6.17}$$

Let us use these representations for obtaining a singular integral equation corresponding to the problem for a cut. In this case, we shall assume that the conditions for stresses are known, while the derivative of the jump in displacements has to be determined.

We obtain the following singular equation:

$$\int_{-l}^{l} \frac{Q(t) + iq(t)}{t - x} dt = \pi p(x), \qquad |x| < l. \tag{6.18}$$

The solution of this equation must satisfy the condition

$$\int_{-l}^{l} g'(t) dt = 0 \tag{6.19}$$

which ensures the uniqueness of displacements. The solution then is unique and has the form

$$g'(x) = -i \frac{\varkappa - 1}{\varkappa + 1} q(x) + \frac{1}{\pi\sqrt{l^2 - x^2}} \left[-\int_{-l}^{l} \frac{\sqrt{l^2 - t^2} p(t)}{t - x} dt - iR \right], \tag{6.20}$$

where

$$R = \frac{\varkappa - 1}{\varkappa + 1} \int_{-l}^{l} q(t) dt.$$

Let us now write the expressions for the functions $\Phi(z)$ and $\Omega(z)$:

$$\Phi(z) = \frac{1}{2\pi\sqrt{z^2 - l^2}} \int_{-l}^{l} \frac{\sqrt{l^2 - t^2} p(t) dt}{t - z} + \frac{1}{2\pi i} \left[\int_{-l}^{l} \frac{q(t) dt}{t - z} + \frac{R}{\sqrt{z^2 - l^2}} \right],$$

$$\Omega(z) = \frac{1}{2\pi\sqrt{z^2 - l^2}} \int_{-l}^{l} \frac{\sqrt{l^2 - t^2} p(t) dt}{t - z} + \frac{1}{2\pi i} \left[- \int_{-l}^{l} \frac{q(t) dt}{t - z} + \frac{R}{\sqrt{z^2 - l^2}} \right]. \tag{6.21}$$

Thus, the problem for a single cut has been completely solved.

Let us now consider a system of N arbitrarily oriented cuts of length $2l_k$ (Fig. 36).

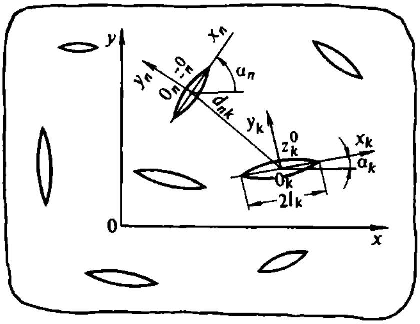

Fig. 36. A system of cuts.

We denote the centres of cuts by O_k and their coordinates by x_k^0 and y_k^0. For each cut, we introduce a local system of coordinates, with the origin at the point O_k, in such a way that the axes x_k are directed along the cut. The edges of the cuts are subjected to the stresses

$$\sigma_k^{\pm} - i\tau_k^{\pm} = p_k(x_k) \pm q_k(x_k), \qquad |x_k| < l_k \qquad (k = 1, 2, \ldots, N).$$

Let us first examine the problem of determining the stresses in an infinite plane with a single cut $|x_k| \leqslant l_k, y_k = 0$. It follows from above that the potentials in the system of coordinates $x_k O_k y_k$ have the form

$$\Phi(z_k) = \frac{1}{2\pi} \int_{-l_k}^{l_k} \frac{Q_k(t)dt}{t - z_k},$$
$$\Psi(z_k) = \frac{1}{2\pi} \int_{-l_k}^{l_k} \left[\frac{\overline{Q_k(t)} - 2i\overline{q_k(t)}}{t - z_k} - \frac{tQ_k(t)}{(t - z_k)^2} \right] dt, \tag{6.22}$$

where $$z_k = x_k + iy_k, Q_k(x) = g'_k(x) - i\frac{2q_k(x)}{\varkappa + 1}.$$

A superposition of these equations gives the following representations:

$$\Phi(z) = \frac{1}{2\pi} \sum_{k=1}^{N} \int_{-l_k}^{l_k} \frac{Q_k(t)dt}{t - z_k},$$
$$\Psi(z) = \frac{1}{2\pi} \sum_{k=1}^{N} e^{-2i\alpha_k} \int_{-l_k}^{l_k} \left[\frac{\overline{Q_k(t)} - 2i\overline{q_k(t)}}{t - z_k} - \frac{\overline{T}_k e^{i\alpha_k}}{(t - z_k)^2} Q_k(t) \right] dt, \tag{6.23}$$

where

$$T_k = te^{i\alpha_k} + z_k^0, \qquad z_k = e^{-i\alpha_k}(z - z_k^0)$$

describes the state of stress for a plane for given discontinuities $g_k(x_k)$ in displacements and $q_k(x_k)$ in stresses over all N intervals.

Let us use the expressions (6.23) for constructing a system of singular equations when the stresses at the edges of the cuts are given. The discontinuities in stresses are precisely satisfied by the structure of the representations (6.23) while the remaining conditions of the equality of stresses can be written as

$$\sigma_n^+ + \sigma_n^- - i(\tau_n^+ + \tau_n^-) = 2p_n(x_n), \qquad |x_n| < l_n \qquad (n = 1, 2, \ldots, N). \quad (6.24)$$

Using the formulas for coordinate transformation, we can write the functions $\Phi_n(z_n)$ and $\Psi_n(z_n)$ in the system of coordinates $x_nO_ny_n$ as follows:

$$\Phi_n(z_n) = \frac{1}{2\pi}\sum_{k=1}^{N}\int_{-l_k}^{l_k}\frac{Q_k(t)dt}{t - z_k},$$

$$\Psi_n(z_n) = \frac{1}{2\pi}\sum_{k=1}^{N} e^{2i\alpha_{nk}}\int_{-l_k}^{l_k}\left[\frac{\overline{Q_k(t)} - 2i\overline{q_k(t)}}{t - z_k} - \frac{(\overline{T}_k - \bar{z}_n^0)e^{i\alpha_k}}{(t - z_k)^2}Q_k(t)\right]dt, \quad (6.25)$$

where

$$z_k = e^{-i\alpha_k}(z_ne^{i\alpha_n} + z_n^0 - z_k^0), \qquad \alpha_{nk} = \alpha_n - \alpha_k.$$

Having determined the stresses on the axis x_n and substituting them into the conditions (6.24), we obtain a system of N singular equations:

$$\sum_{k=1}^{N}\int_{-l_k}^{l_k}\left[Q_k(t)K_{nk}(t, x) + \overline{Q_k(t)}L_{nk}(t, x) + \frac{iq_k(t)}{\overline{T}_k - \overline{X}_n}e^{i(\alpha_k - 2\alpha_n)}\right]dt = \pi p_n(x),$$

$$|x| < l_n \qquad (n = 1, 2, \ldots, N). \quad (6.26)$$

Here

$$K_{nk}(t, x) = \frac{e^{i\alpha_k}}{2}\left(\frac{1}{T_k - X_n} + \frac{e^{-2i\alpha_n}}{\overline{T}_k - \overline{X}_n}\right),$$

$$L_{nk}(t, x) = \frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k - \overline{X}_n} - \frac{T_k - X_n}{(\overline{T}_k - \overline{X}_n)^2}e^{-2i\alpha_n}\right],$$

where

$$X_n = xe^{i\alpha_n} + z_n^0$$

(here and below we omit the subscript in x_n). If the stresses are the same on all sides of the cuts, the right-hand side of Eq. (6.26) is simplified, since $q_k(t) = 0$. We then get

$$\int_{-l_n}^{l_n} \frac{g_n'(t)dt}{t - x} + \sum_{k \neq n} \int_{-l_k}^{l_k} [g'_k(t)K_{nk}(t, x) + \overline{g'_k(t)}\, L_{nk}(t, x)]dt = \pi p_n(x), \tag{6.27}$$

$$|x| < l_n \quad (n = 1, 2, \dots, N).$$

It should be noted that the system of equations (6.26) can be simplified to the following form when the cuts are collinear:

$$\sum_{k=1}^{N} \int_{-l_k}^{l_k} \frac{R_k(t)dt}{t - x + x_k^0 - x_n^0} = \pi p_n(x),\ |x| < l_n \quad (n = 1, 2, \dots, N), \tag{6.28}$$

where

$$R_k(x) = g'_k(x) + i\, \frac{\varkappa - 1}{\varkappa + 1}\, q_k(x).$$

The system of singular integral equations obtained in this way may be solved by the small parameter method (by successive approximations [33]) for sufficiently large distances between cuts.

Let us solve the system (6.27) by the method of mechanical quadratures. To begin with, we reduce all limits of integration to the same interval $[-1, 1]$. For this purpose, we make the following substitution of variables:

$$t = l_n\tau, \qquad x = l_n\xi \qquad (|\tau| < 1, |\xi| < 1). \tag{6.29}$$

The system of equations (6.26) then assumes the following form if we take into account the conditions (6.19):

$$\int_{-1}^{1} \frac{g'_n(\tau)d\tau}{\tau - \xi} + \sum_{k \neq n} l_k \int_{-1}^{1} [g'_k(\tau)K_{nk}(l_k\tau, l_n\xi)$$

$$+ \overline{g'_k(\tau)}L_{nk}(l_k\tau, l_n\xi)]d\tau = \pi p_n(\xi), \qquad |\xi| < 1 \tag{6.30}$$

$$\int_{-1}^{1} g_n'(\tau)d\tau = 0 \qquad (n = 1, 2, \dots, N).$$

Here we retain the above notation for $g_n'(\tau)$ and $p_n(\xi)$.

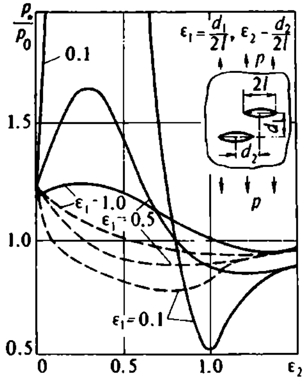

Fig. 37. Two collinear cuts and the value of the critical load.

Using the numerical method described in Sec. 3, Ch. 1, Vol. 1 for solving singular integral equations, we can reduce the above problem to the solution of the following system of algebraic equations:

$$\frac{1}{M}\sum_{m=1}^{M}\sum_{k=1}^{N} l_k[u_k(t_m)K_{nk}(l_kt_m, l_nx_r) + \overline{u_k(t_m)}L_{nk}(l_kt_m, l_nx_r)] = p_n(x_r), \tag{6.31}$$

$$\sum_{m=1}^{M} u_n(t_m) = 0 \qquad (n = 1, 2, \dots, N, r = 1, 2, \dots, M-1),$$

where $g_n'(\xi) = \dfrac{u_n(\xi)}{\sqrt{1-\xi^2}}$. Here, the points t_m and x_r are defined in accordance with the formulas (3.21) and (3.22) of Ch. 1, Vol. 1.

By way of an example, let us consider the problem involving two collinear cuts in a plane (Fig. 37). The loading was reduced just to the stresses σ_y^∞ at infinity. Calculations were performed by choosing 20 points on each cut. Figure 37 shows the values of the critical load (defined according to the brittle fracture theory) for different mutual positions of the cuts, characterized by the parameters $\varepsilon_1 = \dfrac{d_1}{2l}$ and $\varepsilon_2 = \dfrac{d_2}{2l}$. The solid lines correspond to the right-hand tip of the lower cut, while the dashed lines correspond to the left-hand tip.

It should be noted that the equations obtained above can be used for solving problems of bending plates with cuts. However, a rigorous formulation of such

problems is in contradiction with the assumptions forming the basis of the theory of bending of plates, since such problems are essentially three-dimensional, and we can speak of the reliability of the solution only at a certain distance from the end points.

Section 7
The Method of Functionally Invariant Solutions

The apparatus of the theory of functions of a complex variable can be applied for constructing a special class of solutions of the problems in the dynamic theory of elasticity. This class of solutions may be obtained with the help of the so-called functionally invariant solutions of a wave equation.

The functionally invariant solutions are constructed in the following way: we seek a function $\Omega(x, y, t)$ such that any doubly differentiable function[12] $u = f(\Omega)$ is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{7.1}$$

i.e.

$$f''(\Omega)\left[\left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2 - \frac{1}{a^2}\left(\frac{\partial \Omega}{\partial t}\right)^2\right] + f'(\Omega)\left[\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} - \frac{1}{a^2}\frac{\partial^2 \Omega}{\partial t^2}\right] = 0. \tag{7.1'}$$

In view of the arbitrariness of $f(\Omega)$, this gives

$$\begin{gathered}\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 \Omega}{\partial t^2},\\ \left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2 = \frac{1}{a^2}\left(\frac{\partial \Omega}{\partial t}\right)^2.\end{gathered} \tag{7.2}$$

In order that $u = f(\Omega)$ be a solution of the wave equation, the necessary and sufficient condition is that the system of equations (7.2) be satisfied. The validity of the following statement can be proved by direct verification: the total integral of the system (7.2) is an expression which is linear in x, y, and t:

$$\delta = l(\Omega)t + m(\Omega)x + n(\Omega)y - k(\Omega) = 0. \tag{7.3}$$

The coefficients in this equation are related in the following manner:

$$l^2(\Omega) = a^2[m^2(\Omega) + n^2(\Omega)]. \tag{7.4}$$

As a matter of fact, taking the derivatives of an implicitly given function, substi-

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[12] If $\Omega(x, y, t)$ is a complex function, $f(\Omega)$ will be an analytical function.

tuting these into (7.2), and taking into account Eq. (7.4), we find that Eq. (7.3) gives the integral of the system (7.2). Thus, we have proved that the wave equation (7.1) has a class of solutions of the form indicated above. It should be noted that if $f(\Omega)$ is a complex function, the functions $u = \operatorname{Re} f(\Omega)$ and $u = \operatorname{Im} f(\Omega)$ will also be its solutions.

Below, we give the derivatives of the function $f(\Omega)$ with respect to x, y, and t, which we shall be requiring later:

$$\frac{\partial^2 f}{\partial x^2} = f''\left(-\frac{m}{\delta'}\right)^2 + \frac{f'}{\delta'}\frac{\partial}{\partial\Omega}\left(\frac{m^2}{\delta'}\right) = \frac{1}{\delta'}\frac{\partial}{\partial\Omega}\left(f'\frac{m^2}{\delta'}\right),$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{\delta'}\frac{\partial}{\partial\Omega}\left(f'\frac{n^2}{\delta'}\right), \quad \frac{\partial^2 f}{\partial t^2} = \frac{1}{\delta'}\frac{\partial}{\partial\Omega}\left(f'\frac{l^2}{\delta'}\right),$$

$$\frac{\partial^2 f}{\partial y\partial t} = \frac{1}{\delta'}\frac{\partial}{\partial\Omega}\left(f'\frac{nl}{\delta'}\right), \quad \frac{\partial^2 f}{\partial x\partial t} = \frac{1}{\delta'}\frac{\partial}{\partial\Omega}\left(f'\frac{ml}{\delta'}\right), \tag{7.5}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{1}{\delta'}\frac{\partial}{\partial\Omega}\left(f'\frac{mn}{\delta'}\right).$$

The derivatives with respect to Ω are indicated by a prime.

It should be noted that while constructing the solution $u = f(\Omega)$ of the wave equation, it was required that $f(\Omega)$ be differentiable with respect to Ω. Indeed, if for real values of variables the function $\Omega(x, y, t)$ assumes in the plane of complex variable $z = x + iy$ the values that fill a certain domain, it can be assumed that the function $f(\Omega)$ is analytical in this domain. In an implicit form, this assumption is known to contain the Laplace equation for the real and imaginary parts of the function f. The solution $u = f(x + iy)$ is an example of this kind. If, however, the function $f(\Omega)$ assumes a set of values which depend on one real parameter, i.e. which lie on a certain line, the condition of analyticity is not necessary. In this case, in order to get a solution of $f(\Omega)$ in the classical sense, it is sufficient that this function be twice continuously differentiable along this line. The waves $u = f(x \pm at)$ are an example of this kind.

It should be also noted that as shown above the class of the functionally invariant solutions of a wave equation is determined by the structure of the function $\Omega(x, y, t)$ which satisfies the system (7.2) and consequently has the form (7.3) if the condition (7.4) is satisfied. As indicated above, the functions $f(\Omega)$ themselves in this case may be arbitrary twice differentiable (or analytical) functions. It is this property of the solutions which is reflected in the title "method of functionally invariant solutions". This expression indicates certain common properties of groups of solutions of the wave equation.

1. Plane Waves. The simplest form of solution is obtained by assuming l, m, and n as constants and taking the function $k(\Omega)$ equal to Ω. Equation (7.3) then assumes the form

$$\Omega = lt + mx + ny, \tag{7.6}$$

where $l^2 = a^2(m^2 + n^2)$, and the class of solutions of the wave equation is described by functions of the type

$$u = f(lt + mx + ny). \tag{7.7}$$

If l, m, and n are all real numbers, we obtain a solution of the wave equation known as a plane wave. The coefficients l, m, and n may be complex numbers. For $m = 1, n = \pm i$, and $l = 0$, we get a general solution of the Laplace equation, satisfying, of course, the wave equation as well. If, however, all the three coefficients are complex and different from zero, we get an entirely new solution called a complex plane wave [34].

Without any loss of generality, the coefficient l in (7.7) can be put equal to unity. Then, denoting m by $-\theta$, we get $n = \pm\sqrt{a^{-2} - \theta^2}$ and hence the expression (7.7) assumes the form

$$u = f(t - \theta x \pm y\sqrt{a^{-2} - \theta^2}). \tag{7.8}$$

If (7.8) satisfies the Laplace equation, the real and imaginary parts

$$u = \operatorname{Re} f(t - \theta x \pm y\sqrt{a^{-2} - \theta^2}),$$
$$u = \operatorname{Im} f(t - \theta x \pm y\sqrt{a^{-2} - \theta^2})$$

will also be the solutions of Eq. (7.1).

By way of an example, let us consider the reflection of plane waves at the boundary $y = 0$ of an elastic half-space $y \geqslant 0$. We shall designate as a plane longitudinal wave such a solution of the equations

$$\Delta\Phi = \frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2}, \qquad \text{(5.54), Ch. III, Vol. 1,}$$

$$\Delta\Psi = \frac{1}{b^2}\frac{\partial^2\Psi}{\partial t^2}, \qquad \text{(5.55), Ch. III, Vol. 1,}$$

for which

$$\begin{aligned} \Psi(x, y, t) &= 0, \\ \Phi &= f(t - \theta x \pm y\sqrt{a^{-2} - \theta^2}). \end{aligned} \tag{7.9}$$

If in the half-space $y > 0$ we consider the problem of elastic vibrations caused by a plane longitudinal wave, the solution of the type (see Fig. 38)

$$\Phi = f(t - \theta x + y\sqrt{a^{-2} - \theta^2}), \qquad \Psi = 0, \tag{7.10}$$

is called a wave propagating towards the boundary. Apparently, in view of the fact that the coefficients of the function $\Omega(x, y, t)$ are real, we find that θ is a real number: $|\theta| < a^{-1}, \sqrt{a^{-2} - \theta^2} \neq 0$. The wave propagating away from the boundary is described by the relation

$$\Phi = f(t - \theta x - y\sqrt{a^{-2} - \theta^2}), \qquad \Psi = 0. \tag{7.11}$$

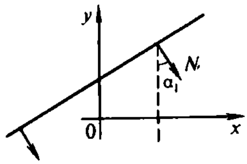

Fig. 38. A wave incident on the boundary.

By a plane transverse wave we mean a solution of Eqs. (1.56), (1.57), Ch. 3, Vol. 1, which can be represented in the form

$$\Phi = 0, \quad \Psi = f(t - \theta x \pm y\sqrt{b^{-2} - \theta^2}). \tag{7.12}$$

The coefficients θ and $\sqrt{b^{-2} - \theta^2}$ are real in this case also. We shall distinguish between transverse waves propagating towards the boundary $y = 0$ of the half-space, for which

$$\Phi = 0, \quad \Psi = f(t - \theta x + y\sqrt{b^{-2} - \theta^2}), \tag{7.13}$$

and the transverse waves propagating from the boundary, for which

$$\Phi = 0, \quad \Psi = f(t - \theta x - y\sqrt{b^{-2} - \theta^2}). \tag{7.14}$$

The geometrical sense in calling these waves as propagating towards or from the boundary is apparent. In the waves propagating towards the boundary, the potential retains a constant value on the planes

$$t - \theta x + y\sqrt{a^{-2} - \theta^2} = \text{const} \quad \text{(longitudinal wave)}, \tag{7.14'}$$

$$t - \theta x + y\sqrt{b^{-2} - \theta^2} = \text{const} \quad \text{(transverse wave)}, \tag{7.14''}$$

which are displaced with increasing t in such a way that the direction N of their motion, characterized by the direction cosines

$$\begin{aligned} &\cos(N, x) = \theta a, \quad \cos(N, y) = -\sqrt{1 - \theta^2 a^2} \quad \text{(longitudinal wave)},\\ &\cos(N, x) = \theta b, \quad \cos(N, y) = -\sqrt{1 - \theta^2 b^2} \quad \text{(transverse wave)}, \end{aligned} \tag{7.15}$$

forms an obtuse angle with the y-axis. On the other hand, the direction cosines of the normal N_1 to the wavefronts in the case of waves propagating from the boundary have the form

$$\begin{aligned} &\cos(N_1, x) = \theta a, \quad \cos(N_1, y) = \sqrt{1 - \theta^2 a^2} \quad \text{(longitudinal wave)},\\ &\cos(N_1, x) = \theta b, \quad \cos(N_1, y) = \sqrt{1 - \theta^2 b^2} \quad \text{(transverse wave)}, \end{aligned} \tag{7.16}$$

and the normal forms an acute angle with the y-axis (Fig. 39).

Suppose that there are no stresses at the boundary of the half-space. In this case, we get, for $y = 0$,

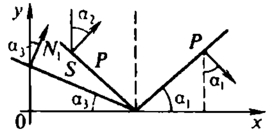

Fig. 39. Incident longitudinal wave and reflected waves (S — transverse wave, P — longitudinal wave).

$$\begin{aligned} \tau_{xy} &= \rho b^2 \left[2\frac{\partial^2\Phi}{\partial x\,\partial y} + \frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x^2}\right] = 0, \\ \sigma_y &= \rho\left[a^2\frac{\partial^2\Phi}{\partial y^2} + (a^2 - 2b^2)\frac{\partial^2\Phi}{\partial x^2} - 2b^2\frac{\partial^2\Psi}{\partial x\,\partial y}\right] = 0. \end{aligned} \tag{7.17}$$

It can easily be verified by direct substitution that neither the wave propagating towards the boundary, nor the one propagating from it can satisfy the boundary conditions separately. Hence it is natural and physically justified to seek a solution of the problem for a half-space in the form of a sum of different types of waves. In view of the linearity of equations of dynamics of an elastic body, such a treatment is justified. Besides, it should be noted that the initial conditions have been taken into account by appropriately choosing the direction of propagation of the wave.

To begin with, let us suppose that a longitudinal plane wave, described by the potentials of the type (7.10), propagates towards the boundary $y = 0$. We shall assign the index "1" to this wave, and shall seek the waves reflected at the boundary in the form of a sum of longitudinal and transverse waves propagating from the boundary:

$$\begin{aligned} \Phi_2 &= Af(t - \theta x - y\sqrt{a^{-2} - \theta^2}), \\ \Psi_2 &= Bf(t - \theta x - y\sqrt{b^{-2} - \theta^2}), \end{aligned} \tag{7.18}$$

Here, A and B are unknown constants called the wave amplitudes or the reflection coefficients. These constants must be determined from the conditions (7.17) after substituting the expressions $\Phi = \Phi_1 + \Phi_2$ and $\Psi = \Psi_1 + \Psi_2 = \Psi_2$ in them. As a result of substitution, we get

$$\begin{aligned} &[(1 - 2\theta^2b^2)(1 + A) - 2\theta b^2B\sqrt{b^{-2} - \theta^2}]f''(t - \theta x) = 0, \\ &[-2\theta(1 - A)\sqrt{a^{-2} - \theta^2} + (b^{-2} - 2\theta^2)B]f''(t - \theta x) = 0. \end{aligned} \tag{7.19}$$

Since f is an arbitrary function, the expressions in the square brackets are equal to zero, and we get the following equations for A and B from the system (7.19):

$$
\begin{aligned}
A &= \frac{1}{R(\theta)}[-(2\theta^2 - b^{-2})^2 + 4\theta^2\sqrt{a^{-2} - \theta^2}\sqrt{b^{-2} - \theta^2}],\\
B &= \frac{1}{R(\theta)}[-4\theta(2\theta^2 - b^{-2})\sqrt{a^{-2} - \theta^2}],
\end{aligned}
\tag{7.20}
$$

where

$$R(\theta) = (2\theta^2 - b^{-2}) + 4\theta^2\sqrt{a^{-2} - \theta^2}\sqrt{b^{-2} - \theta^2} > 0.$$

Consequently, we obtain the required expressions for the potentials of the reflected waves with the help of Eqs. (7.18) and (7.20). It should be remarked that while finding a solution of the problem about the reflection of a longitudinal plane wave at the free boundary of a half-space, it was assumed that the reflected waves are described by the same function $f(\Omega)$ as the incident waves. This function describes the profile of an incident wave. It can be seen from the solution (7.20) that there are also reflected waves having the same profile. If the observer (instrument) is placed at a certain point (x, y) of the half-plane so that the incident longitudinal wave and the reflected longitudinal and transverse waves pass through this point at instants t_{1p}, t_{2p}, and t_{2s} respectively, the observer can register the change in perturbation (displacement, stress, or strain) with time for each of these waves according to the law $f(t)$. The effect of the amplitudes A and B, appearing in the scale factor for the ordinate axis in the $f = f(t)$ curve, is manifested for the reflected waves. In other words, $f(t)$ may be called the theoretical oscillogram of a perturbation or a wave.

The preservation or distortion of the "shape" of a wave (or the shape of a theoretical oscillogram) has a considerable importance in the applications of the dynamic theory of elasticity.

We shall mention some more geometrical results following from the formulas for the potential of the incident and the reflected waves. We denote the angle of incidence of the wave on the boundary by α_1, the angle between the normal to the surface,

$$t - \theta x + y\sqrt{a^{-2} - \theta^2} = \text{const}$$

(which may be called the wavefront at the moment t for an appropriate choice of the constant on the right-hand side) and the direction of negative values of y; and the angles of reflection, α_2 and α_3, are the angles formed by the normals to the surfaces,

$$t - \theta x - y\sqrt{a^{-2} - \theta^2} = \text{const}, \qquad t - \theta x - y\sqrt{b^{-2} - \theta^2} = \text{const},$$

with the positive direction of the y-axis. Using this notation, we get from the formulas for the potential: (1) $\alpha_1 = \alpha_3$, i.e. the angle of incidence of the longitudinal wave is equal to the angle of reflection, and (2) $\sin\alpha_1/\sin\alpha_3 = a/b$, i.e. the ratio of the sines of the angle of incidence of the longitudinal wave and of the angle of reflection of the transverse wave is equal to the ratio of the velocities of propagation of longitudinal and transverse waves (see Fig. 39). From formula (7.16) it follows that $\theta = (\sin\alpha_1)/a$.

Let us now consider a plane transverse wave, incident on the boundary $y = 0$. If $|\theta| < a^{-1}$, this problem does not pose any new difficulties in comparison with the

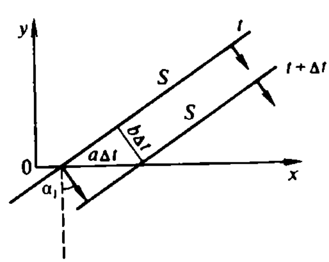

Fig. 40. Transition to the case of total internal reflection.

previous one, and can be solved in a similar manner. The incident wave is described by the potentials

$$\Phi_1 = 0, \qquad \Psi_1 = f(t - \theta x + y\sqrt{b^{-2} - \theta^2}). \tag{7.21}$$

The solution for the reffected waves may be sought in the form

$$\begin{aligned} \Phi_2 &= Cf(t - \theta x - y\sqrt{a^{-2} - \theta^2}), \\ \Psi_2 &= Df(t - \theta x - y\sqrt{b^{-2} - \theta^2}, \end{aligned} \tag{7.22}$$

where C and D are constants having the meaning of the reflection coefficients (amplitudes of reflected waves).

Substituting $\Phi = \Phi_2$, $\Psi = \Psi_1 + \Psi_2$ into (7.17), we get

$$\begin{aligned} &[(1 - 2\theta^2 b^2)C + 2\theta b^2(1 - D)\sqrt{b^{-2} - \theta^2}]f''(t - \theta x) = 0, \\ &[2\theta C\sqrt{a^{-2} - \theta^2} + (1 + D)(b^{-2} - 2\theta^2)]f''(t - \theta x) = 0. \end{aligned} \tag{7.23}$$

From this, assuming that $f''(t) \neq 0$, we get

$$\begin{aligned} C &= \frac{1}{R(\theta)}[4\theta(2\theta^2 - b^{-2})\sqrt{b^{-2} - \theta^2}], \\ D &= \frac{1}{R(\theta)}[-(2\theta^2 - b^{-2})^2 + 4\theta^2\sqrt{b^{-2} - \theta^2}\sqrt{a^{-2} - \theta^2}]. \end{aligned} \tag{7.24}$$

Let us consider the restriction $|\theta| < a^{-1}$, imposed while describing the reflection of the transverse waves. Formally, this restriction follows from the real nature of coefficients in the expression for $\Omega(x, y, t)$, which is a necessary condition for the simplest plane wave (7.7). From a mechanical point of view, this means that $|\theta| = |\theta_0| = a^{-1}$ defines the angle of incidence of a plane transverse wave, at which the total internal reflection takes place. From formulas (7.15), we can find a relation between the angle α_1 of incidence of a plane transverse wave on the boundary $y = 0$ and θ_0 (assuming, for the sake of definiteness, that the direction of propagation of the wave forms an acute angle with the positive direction of the x-axis, and hence $\theta > 0$):

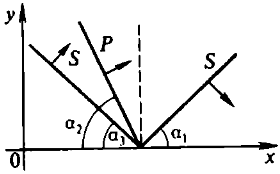

Fig. 41. Reflection of waves in the case of an incident transverse wave.

$$\sin\alpha_1 = \theta_0 b = \frac{b}{a}\,. \tag{7.25}$$

It can be easily seen from Fig. 40 that Eq. (7.25) defines the value α_1 of the angle for which the velocity of propagation of transverse waves (along the normal to the wavefront, a wave travels the distance $b\Delta t$ during the time Δt) along the boundary is equal to the velocity of longitudinal waves (it follows from Fig. 40 that a longitudinal wave will travel a distance of $a\Delta t$ during this time). For angles of incidence $\alpha_1 > \arcsin(b/a)$, the total internal reflection of transverse waves will take place. The longitudinal perturbations appearing at the points on the surface $y = 0$ as a result of the incidence of a transverse wave on this surface will overtake this wave. This property can be interpreted as follows: the sine of the angle of reflection of a longitudinal wave, calculated according to the sine law, $\sin\alpha_2 = \theta a$, turns out to be greater than unity, and hence there is no real angle of reflection of a longitudinal wave in the usual sense. Thus, the solution of the problem of reflection, represented by the formulas (7.22) and (7.24), is valid only for $|\theta| < a^{-1}$, i.e. for such angles of incidence of the wave that are less than the angle of the total internal reflection: $\sin\alpha_1 < b/a$ (Fig. 41).

Let us now consider the case of total internal reflection. It is natural to assume that even for $|\theta| < a^{-1}$, the solution may be sought as a sum of plane waves. But in this case, we do not have to require that the coefficients be real in the expression for $\Omega(x, y, t)$ in the solution for the longitudinal potential, since the coefficients for a plane longitudinal wave may become complex for $|\theta| < a^{-1}$. It can be shown that the following problem is solvable for $|\theta| > a^{-1}$: for the given incident transverse wave (7.21), find the reflected longitudinal and transverse waves with bounded displacements, such that the resultant perturbations satisfy the given boundary conditions. In this case, the solvability condition for this problem is that the vector $\mathbf{w} = \operatorname{grad}\Phi_2$ be bounded, which is the same as the condition that $f'(x)$ be bounded.

Let us consider two auxiliary cases which will help us in constructing a general solution for the reflection problem. Suppose that the function $f = f_1$ in (7.21) is complex and represents the values of a certain function of a complex variable on the real axis. This function is regular in the half-plane $y > 0$ and has a bounded derivative on it. The solution of the problem is sought in the form

$$\begin{aligned} \Phi_2 &= C_1 f_1(t - \theta x + iy\sqrt{\theta^2 - a^{-2}}), \\ \Psi_2 &= D_1 f_1(t - \theta x - y\sqrt{b^{-2} - \theta^2}), \end{aligned} \tag{7.26}$$

where C_1 and D_1 are constants determined from the homogeneous boundary conditions (7.17). Comparing (7.22) and (7.26), it can be easily seen that C_1 and D_1 are obtained from C and D in (7.24) by simply replacing the radical $\sqrt{a^{-2} - \theta^2}$ by $- i\sqrt{\theta^2 - a^{-2}}$.

In an exactly similar way, we can consider the second auxiliary problem, when $f = f_2$ is the boundary value of a function which is analytical in the half-plane $y < 0$. In this case, the reflected waves are described with the help of the potentials

$$\Phi_2 = C_2 f_2(t - \theta x - iy\sqrt{\theta^2 - a^{-2}}),$$
$$\Psi_2 = D_2 f_2(t - \theta x - y\sqrt{b^{-2} - \theta^2}), \tag{7.27}$$

and the radical $\sqrt{a^{-2} - \theta^2}$ in (7.24) should be replaced by $i\sqrt{\theta^2 - a^{-2}}$.

The auxiliary solutions (7.26) and (7.27) can be used for solving a problem in the general case. We proceed from the fact that a certain function $f(x)$ of a real variable, which has a continuous second derivative satisfying the inequality

$$|x^2 f''(x)| < M,$$

can be represented in the form

$$f(x) = \frac{1}{2}[f_1(x) + f_2(x)], \tag{7.28}$$

where $f_1(x)$ is the boundary value of a function of a complex variable, which is defined and analytical on the half-plane $y > 0$ and has a bounded first derivative on it, while $f_2(x)$ is the boundary value of a function which is analytical on the half-plane $y < 0$ and has a bounded first derivative on it[13]. The conditions imposed on $f(x)$ lead to the boundedness of $f_1'(x)$ and $f_2'(x)$—the derivatives of the boundary values of the above-mentioned analytical functions. While solving the general problem on the reflection of the transverse wave

$$\Phi_1 = 0, \qquad \Psi_1 = f(t - \theta x + y\sqrt{b^{-2} - \theta^2}),$$

where f is an arbitrary function for $b^{-1} > \theta > a^{-1}$, we can represent Ψ_1 as a sum of two components with the help of (7.28):

$$\Psi_1 = \frac{1}{2}[f_1(t - \theta x + y\sqrt{b^{-2} - \theta^2}) + f_2(t - \theta x + y\sqrt{b^{-2} - \theta^2})]. \tag{7.29}$$

We solve this problem separately for each component. This leads to formulas describing the total field of the incident and the reflected wave:

$$\Phi = \frac{1}{2}[C_1 f_1(t - \theta x + iy\sqrt{\theta^2 - a^{-2}}) + C_2 f_2(t - \theta x - iy\sqrt{\theta^2 - a^{-2}})], \tag{7.30}$$

---

[13] The validity of this fact follows from the construction of analytical functions for $y > 0$ and $y < 0$ according to their real part, which is the same for $y = 0$ and is equal to $f(x)$. It follows from the Schwarz symmetry principle (see [35]) that $f_1(x)$ and $f_2(x)$ are conjugate functions.

$$\Psi = \frac{1}{2}[f_1(t - \theta x + y\sqrt{b^{-2} - \theta^2}) + f_2(t - \theta x + y\sqrt{b^{-2} - \theta^2})]$$
$$+ \frac{1}{2}[D_1 f_1(t - \theta x - y\sqrt{b^{-2} - \theta^2})$$
$$+ D_2 f_2(t - \theta x - y\sqrt{b^{-2} - \theta^2})]. \qquad (7.31)$$

Since the constants D_1 and D_2, as well as C_1 and C_2, are complex conjugates, the results (7.30) and (7.31) may be written in the following form:

$$\Phi = \mathrm{Re}\{C_1 f_1(t - \theta x + iy\sqrt{\theta^2 - a^{-2}})\},$$

$$\Psi = \mathrm{Re}\{f_1(t - \theta x + y\sqrt{b^{-2} - \theta^2}) + D_1 f_1(t - \theta x - y\sqrt{b^{-2} - \theta^2})\}. \qquad (7.32)$$

Before concluding this subsection, we shall mention a specific form of the function f, which is frequently encountered in applications:

$$f_1(t - \theta x + y\sqrt{b^{-2} - \theta^2}) = Ae^{i\omega(t - \theta x + y\sqrt{b^{-2} - \theta^2})} \qquad (7.33)$$

or

$$f(t - \theta x + y\sqrt{b^{-2} - \theta^2}) = A\cos\omega(t - \theta x + y\sqrt{b^{-2} - \theta^2}).$$

This is the case of the so-called steady-state vibrations (with the frequency ω). Here, A is the amplitude of the wave, and $\Omega(x, y, t) = t - \theta x + y\sqrt{b^{-2} - \theta^2}$ is the phase. In this case, the potential of the reflected transverse wave has the form

$$\Psi_2 = \mathrm{Re}\{D_1 Ae^{i\omega(t - \theta x - y\sqrt{b^{-2} - \theta^2})}\}. \qquad (7.34)$$

However, the modulus of the coefficient D_1 of the reflected transverse wave is equal to unity (see (7.24), where D is equal to D_1 if we replace $\sqrt{a^{-2} - \theta^2}$ by $-i\sqrt{a^{-2} - \theta^2}$. Hence the amplitude of the reflected waves is equal to the amplitude of the incident wave. The energies of both waves will consequently be identical, which justifies the term "total internal reflection". By finding the potential of the reflected longitudinal wave, we obtain

$$\Phi = e^{-\omega y\sqrt{\theta^2 - a^{-2}}}\,\mathrm{Re}\{C_1 e^{i\omega(t - \theta x)}\}. \qquad (7.35)$$

This means that the longitudinal vibrations will be damped with increasing y (as we move away from the boundary of the half-plane).

We shall now seek a solution for the case when the longitudinal and transverse perturbations are complex waves, and there are no stresses at the boundary of the half-plane. For potentials, we choose such functions of a complex variable, whose derivative vanishes at infinity. We then get a solution in which the displacements tend to zero at infinity. For this reason, waves of such type are called surface waves.

Suppose that the longitudinal potential Φ is given by the formula

$$\Phi = f_1(t - \theta x + i\sqrt{\theta^2 - a^{-2}}\,y), \qquad (7.36)$$

where θ is a real number, $|\theta| > b^{-1}$, f_1 is a function of a complex variable, which is the boundary value of a function analytical in the upper half-plane, and $|f'_1| < M$.

The potential Ψ is sought in the form

$$\Psi = Af_1(t - \theta x + i\sqrt{\theta^2 - b^{-2}}y). \tag{7.37}$$

Then, substituting these potentials into the conditions (7.17), we get

$$\begin{gathered}[(1 - 2b^2\theta^2) + 2ib^2\theta\sqrt{\theta^2 - b^{-2}}A]f''(t - \theta x) = 0,\\ [-2i\theta\sqrt{\theta^2 - a^{-2}} + (b^{-2} - 2\theta^2)A]f''(t - \theta x) = 0.\end{gathered} \tag{7.38}$$

This system has a non-zero solution when its determinant vanishes:

$$\Delta = b^2[(2\theta^2 - b^{-2})^2 - 4\theta^2\sqrt{\theta^2 - a^{-2}}\sqrt{\theta^2 - b^{-2}}] = 0. \tag{7.39}$$

This equation is called Rayleigh's equation. It can be shown [34] that this equation has one real positive root, lying in the interval $b^{-1} < \theta < \infty$, while the other root is modulo equal to it, but has a negative sign. We denote the positive root by c^{-1} and substitute it for θ in all the expressions. The final results are of the form

$$\begin{gathered}\Phi = f_1\left(t \mp \frac{x}{c} + i\sqrt{c^{-2} - a^{-2}}y\right),\\ \Psi = Af_1\left(t \mp \frac{x}{c} + i\sqrt{c^{-2} - b^{-2}}y\right),\end{gathered} \tag{7.40}$$

where

$$A = \mp 2ic^{-1}\sqrt{c^{-2} - a^{-2}}(2c^{-2} - b^{-2})^{-1} = (2c^{-2} - b^{-2})\left(\pm 2ic^{-1}\sqrt{c^{-2} - b^{-2}}\right)^{-1}.$$

In a similar way, by considering the function f_2 which is the boundary value of a function analytical in the lower half-plane, we arrive at the formulas

$$\begin{gathered}\Phi = f_2\left(t \mp \frac{x}{c} - i\sqrt{c^{-2} - a^{-2}}y\right),\\ \Psi = Bf_2\left(t \mp \frac{x}{c} - i\sqrt{c^{-2} - b^{-2}}y\right),\end{gathered} \tag{7.41}$$

where

$$B = \pm 2ic^{-1}\sqrt{c^{-2} - a^{-2}}(2c^{-2} - b^{-2})^{-1} = (2c^{-2} - b^{-2})(\mp 2ic^{-1}\sqrt{c^{-2} - b^{-2}})^{-1}.$$

If the functions f_1 and f_2 are conjugate, the half-sum of the respective solutions results in a real function which we are seeking:

$$\begin{gathered}\Phi = \mathrm{Re}\left\{f_1\left(t \mp \frac{x}{c} + i\sqrt{c^{-2} - a^{-2}}y\right)\right\},\\ \Psi = \mathrm{Re}\left\{Af_1\left(t \mp \frac{x}{c} + i\sqrt{c^{-2} - b^{-2}}y\right)\right\}.\end{gathered} \tag{7.42}$$

This type of waves are named after Rayleigh, who was the first to establish their existence.

It follows from formulas (7.42) that the entire pattern of motion is displaced along the x-axis with the velocity c, remaining unchanged in a moving coordinate system. The quantity c is called Rayleigh's velocity.

2. Homogeneous Solutions. Let us consider in detail the case when the function $k(\Omega)$ in Eq. (7.3) is identically equal to zero. Then, if $l(\Omega) \neq 0$, we get $n^2(\Omega)/l^2(\Omega) = a^{-2} - \theta^2$ by dividing both sides of Eq. (7.3) by $l(\Omega)$, and denoting $m(\Omega)/l(\Omega)$ by $-\theta$. Consequently, Eq. (7.3) can be written as follows:

$$t - \theta x + \sqrt{a^{-2} - \theta^2} y = 0. \tag{7.43}$$

Thus, we should now write $f(\Omega)$ instead of $f(\theta)$.

Equation (7.43) may be rewritten in the form

$$1 - \theta\xi + \sqrt{a^{-2} - \theta^2}\eta = 0 \qquad \left(\xi = \frac{x}{t}, \eta = \frac{y}{t}\right). \tag{7.44}$$

Consequently, θ is determined from (7.44) as a function of only two arguments ξ and η. In this case, the solutions $f(\theta)$, constructed for Eq. (7.1), will be functions of the arguments ξ and η. In other words, they will be zero-dimensional homogeneous functions of t, x, and y. It is well known [36] that s-dimensional homogeneous functions of the variables t, x, and y are defined by the relation

$$u(kt, kx, ky) = k^s u(t, x, y).$$

The corresponding solutions of the wave equation will be called the s-dimensional homogeneous solutions. Thus, any twice differentiable (analytical, if complex) function $f(\theta)$ is a zero-dimensional solution of the wave equation (7.1), if θ is defined by Eq. (7.43). Conversely, it can be shown [34] that any zero-dimensional homogeneous solution of the wave equation may be written in the form $u = f(\theta)$, where θ is a solution of Eq. (7.43).

Let us analyze Eq. (7.44) in greater detail. In order to isolate the single-valued branch of the radical $\sqrt{a^{-2} - \theta^2}$, we make a cut in the complex plane θ along the real axis $(-a^{-1}, a^{-1})$, and fix the branch of the radical $\sqrt{a^{-2} - \theta^2}$ by the condition $\sqrt{a^{-2} - \theta^2} > 0$ for $\theta = ib$, where $b > 0$. Solving Eq. (7.44) for θ, we get

$$\theta = \frac{\xi - i\eta\sqrt{1 - a^{-2}(\xi^2 + \eta^2)}}{\xi^2 + \eta^2}. \tag{7.45}$$

In view of the branch of the radical $\sqrt{a^{-2} - \theta^2}$ chosen by us, we should take its arithmetic value $[1 - a^{-2}(\xi^2 + \eta^2)]^{1/2}$ in this expression. Under this condition, Eq. (7.44) maps the interior of the circle $\xi^2 + \eta^2 \leqslant a^2$ onto the plane θ with a cut (a^{-1}, a^{-1}). In this case, the upper ($\eta > 0$) and the lower ($\eta < 0$) part of the interior of the circle $\xi^2 + \eta^2 \leqslant a^2$ are mapped onto the lower and the upper half-plane θ respectively. The point $\xi = \eta = 0$ is mapped onto an infinitely removed point on the plane θ, while the segments $\eta = 0$, $0 < \xi < a$ and $\eta = 0$, $-a < \xi < 0$ are mapped onto the real semiaxes $\theta > a^{-1}$ and $\theta < a^{-1}$ respectively. The circle $\xi^2 + \eta^2 = a^2$ is mapped onto the cut $(-a^{-1}, a^{-1})$. The corresponding points of the circle $\xi^2 + \eta^2 \leqslant a^2$ and the plane θ are shown in Figs. 42 and 43.

Let $f(\theta)$ be a single-valued analytical function in the plane θ with a cut $(-a^{-1}, a^{-1})$. The solution of Eq. (7.1) is taken in the form $u = \mathrm{Re} f(\theta)$. This solution is de-

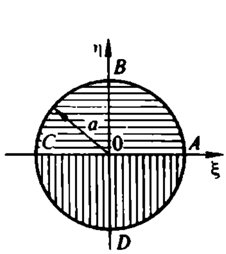

Fig. 42. Domain of variation of progressive variables.

fined inside the circle $\xi^2 + \eta^2 < a^2$. We shall show that it can be continuously extended to the exterior of the circle $\xi^2 + \eta^2 > a^2$. Indeed, let us consider Eq. (7.36) once again. If $\xi^2 + \eta^2 > a^2$, the solution of this equation may be written in the form

$$\theta = \frac{\xi \pm \eta\sqrt{a^{-2}(\xi^2 + \eta^2) - 1}}{\xi^2 + \eta^2}. \tag{7.46}$$

It can be seen from this equation that θ assumes real values for $\xi^2 + \eta^2 > a^2$. For $\theta = \text{const}$, this equation is the equation of the tangent to the circle $\xi^2 + \eta^2 = a^2$. Consequently, θ remains constant on the tangent to the circle, and the point of tangency divides the tangent into two semitangents: I in the counterclockwise direction from the point of tangency (its equation can be obtained from (7.46) by choosing the plus sign for the radical), and II in the clockwise direction (described by Eq. (7.46) with the minus sign for the radical). These semitangents are shown in Figs. 44 and 45 respectively. Assuming that θ remains constant on these semitangents (its values being the same as at the corresponding points of tangency), we can uniquely map the exterior of the circle $\xi^2 + \eta^2 > a^2$ onto the segment $(-a^{-1}, a^{-1})$ in the plane θ (since each of the semitangents I and II is mapped onto a point of this cut). Thus, we can continuously extend the solutions through the boundary $\xi^2 + \eta^2 = a^2$ to the exterior of the circle $\xi^2 + \eta^2 > a^2$, while the value of the solution $u = \operatorname{Re} f(\theta)$ remains constant along the semitangents I or II. Moreover, if we

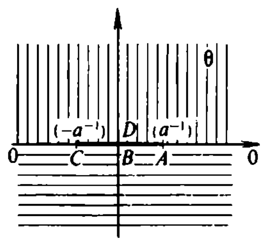

Fig. 43. Points on a complex plane with a cut.

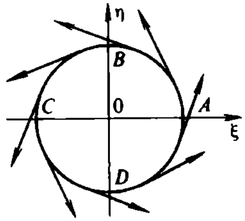

Fig. 44. Semitangents in anticlockwise direction.

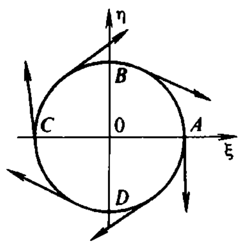

Fig. 45. Semitangents in clockwise direction.

split the solution $u = \mathrm{Re} f(\theta)$ into two real components $u = u_1(\theta) + u_2(\theta)$, where $u_1(\theta)$ and $u_2(\theta)$ are continued along the semitangents I and II to the exterior of the circle $\xi^2 + \eta^2 > a^2$, we once again get a real solution of Eq. (7.1), which is continuously extended to the exterior of the circle through the boundary. Thus, we essentially get an infinite set of various ways of extending a solution. In all these cases, the solution remains continuous as we cross the boundary of the circle. In specific problems, the method of extending a solution is chosen according to the motion of the wavefront.

The above analysis is valid for the case when the solution is considered in the entire space. Let us now turn to the half-space $y \geqslant 0$ (in other words, we consider only the half-plane $y \geqslant 0$ in the plane (ξ, η)). Suppose that the solution of Eq. (7.1) inside the semicircle $\xi^2+\eta^2 \leqslant a^2$, $\eta \geqslant 0$ is taken in the form $u = \mathrm{Re} f(\theta)$, and that this solution vanishes on a certain arc AB of the semicircle (Fig. 46). Then, as shown in Fig. 45, the semitangents I (for $\xi < 0$) and II (for $\xi > 0$) can be used conveniently in many cases to obtain a unique continuation of the solution through the arc of the circle. It follows from this continuation that the solution is identically equal to zero outside the bounded region $A_1ABB_1OA_1$.

Instead of θ, we can choose another complex variable z in Eqs. (7.43) and (7.36), the two variables being connected through an analytical relation. We shall indicate a fairly convenient functional dependence of this kind. Suppose that z and θ are connected through the relation

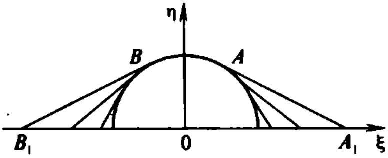

Fig. 46. Domain of perturbed state in variables ξ and η.

$$\theta = \frac{1}{2a}\left(z + \frac{1}{z}\right). \tag{7.47}$$

Here, the plane θ with the cut $(-a^{-1}, a^{-1})$ is transformed into a circle $|z| \leqslant 1$. For such a choice of the branch of the radical, the following equality is valid:

$$\sqrt{a^{-2} - \theta^2} = \frac{i}{2a}\left(z - \frac{1}{z}\right). \tag{7.48}$$

In this case, Eq. (7.36) can be written in the form

$$1 - \frac{1}{2a}\left(z + \frac{1}{z}\right)\xi + \frac{i}{2a}\left(z - \frac{1}{z}\right)\eta = 0$$

or

$$1 - \frac{z}{2a}(\xi - i\eta) - \frac{1}{2az}(\xi + i\eta) = 0. \tag{7.49}$$

In the polar system of coordinates, this equation can be rewritten as

$$\rho e^{-i\varphi} z^2 - 2za + \rho e^{i\varphi} = 0.$$

An obvious solution of this equation is $z = r\exp(i\varphi)$, where r is determined from the quadratic equation and is given by

$$r = \frac{\rho}{a + \sqrt{a^2 - \rho^2}}.$$

Thus, to each point of the circle $\xi^2 + \eta^2 \leqslant a^2$, there is a corresponding point of a circle of unit radius in the z-plane. Formulas (7.47) and (7.49) are especially convenient for considering the dynamic problems of wedge-shaped regions.

It should be noted that the solutions obtained in this way, which remain continuous upon crossing the boundary of the circle $\xi^2 + \eta^2 = a^2$, may have discontinuous normal derivatives upon crossing the boundary of this circle. Moreover, although these solutions are continuous along with all their derivatives inside the circle (in view of the fact that $f(\theta)$ is an analytical function), not only the derivatives but even the solutions themselves may have discontinuities, including infinite discontinuities, outside the circle $\xi^2 + \eta^2 \geqslant a^2$. This fact can be easily understood

if we note that the circles

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 = a^2 \quad (\text{i.e.} \xi^2 + \eta^2 = a^2)$$

and their tangents

$$t - \theta x + \sqrt{a^{-2} - \theta^2}\, y = 0 \qquad (\theta = \text{const}, \ -a^{-1} < \theta < a^{-1})$$

are the cross sections of characteristic surfaces by the planes t = const in the space x, y, t. The characteristic surfaces for a two-dimensional wave equation in the space x, y, t are cones with axes parallel to the t-axis, and their tangents are planes (see, for example, [37]).

The above basic ideas of application of the theory of functions of a complex variable to the solution of the wave equation (7.1) are widely used in the problems of propagation of oscillations, connected with the solution of one wave equation or a system of wave equations.

Section 8
Axially Symmetric Self-modeling Dynamic Problem for a Half-space with Mixed Mobile Boundary Conditions

We shall now present the solution of one class of mixed self-modeling dynamic problems for a half-space[14] [38]. We shall assume that the following conditions are satisfied on the entire surface $z = 0$:

$$\tau_{z\varphi}(r, 0, t) = \tau_{rz}(r, 0, t) = 0. \tag{8.1}$$

Moreover, the displacements $\sigma_z(r/t)$ are given on one part of the boundary, while on the other part, the velocities $v_z(r/t)$ are given. The line of demarcation of the boundary condition is a circle moving with a constant velocity v. The problem is then self-modeling and axially symmetric. Besides, the stresses and velocities are found to be zero-dimensional homogeneous functions.

In order to solve the formulated axially symmetric problem, we use the formulas (5.66), Ch. 3, Vol. 1, establishing a connection between the solutions of axially symmetric problems and the plane problems. In view of the homogeneity of stresses and velocities, the representation for the plane solution is taken in a form defined by the method of functionally invariant solutions.

It follows from the above that the required solution (for velocities and stresses) may be sought in the form

$$v_{(k)r}(r, z, t) = \text{Re} \int_{-\pi}^{\pi} V_{(k)x}(\theta_k) \cos\varphi \, d\varphi,$$

$$v_{(k)z}(r, z, t) = \text{Re} \int_{-\pi}^{\pi} V_{(k)z}(\theta_k) d\varphi,$$

[14] Under specific type of conditions, this problem is found to be equivalent to the problem of expansion of a disk-shaped cut in space.

$$\tau_{(k)rz}(r, z, t) = \mathrm{Re} \int_{-\pi}^{\pi} \Sigma_{(k)xz}(\theta_k)\cos\varphi\, d\varphi, \tag{8.2}$$

$$\sigma_{(k)z}(r, z, t) = \mathrm{Re} \int_{-\pi}^{\pi} \Sigma_{(k)z}(\theta_k) d\varphi,$$

where the index k assumes the values 1 or 2 respectively for the longitudinal and the transverse component of displacements and the stresses defined by them, while θ_k is defined by the formula[15]

$$\theta_k = \frac{a_k t r\cos\varphi + iz\sqrt{a_k^2 t^2 - r^2\cos^2\varphi - z^2}}{a_k(r^2\cos^2\varphi + z^2)}, \tag{8.3}$$

which represents Eqs. (7.14′) and (7.14″) in which we have switched over to the polar system of coordinates and, besides, have replaced Y by $-z$ (since we employed the axes X and Y in the plane problem, the role of the Y-axis in the three-dimensional case is played by z with the opposite sign for the case of axial symmetry).

The representations (8.2) follow from the fact that the displacements and velocities (as mentioned in Sec. 7) may be expressed in terms of analytical functions as follows:

$$v_{(k)i}(X, z, t) = \mathrm{Re}\, V_{(k)j}(\theta_k), \tag{8.4}$$

$$\sigma_{(k)ij}(X, z, t) = \mathrm{Re}\Sigma_{(k)ij}(\theta_k).$$

The functions introduced above are analytical in the half-plane $z > 0$ ($\mathrm{Im}\theta_k > 0$). We can extend these functions to the lower half-plane $z < 0$ ($\mathrm{Im}\theta_k < 0$) as follows:

$$\overline{V_{(k)j}(\theta_k)} = V_{(k)j}(\bar{\theta}_k), \qquad \overline{\Sigma_{(k)ij}(\theta_k)} = \Sigma_{(k)ij}(\bar{\theta}_k). \tag{8.5}$$

In view of the condition (8.1) that the shearing stresses be equal to zero, the solution of the plane problem must satisfy the condition

$$\tau_{Xz}(X, 0, t) = 0. \tag{8.6}$$

Following [39], we shall now give the relations between the analytical functions introduced above, which are valid on the plane $z = 0$. It should be noted that in this case $\theta = \theta_1 = \theta_2 = t/x$. In order to derive these relations, we shall use the boundary conditions (8.6), as well as Eqs. (5.53) and (5.54), Ch. 3, Vol. 1, and the relations between strains and stresses, and displacements. We introduce a function

$$V_z(\theta) = V_{1z}(\theta) + V_{2z}(\theta), \tag{8.7}$$

in terms of which all the functions introduced above can be represented as follows:

$$V_{1X}'(\theta) = \frac{\theta(1 - a_2^2\theta^2)}{\sqrt{a_1^{-2} - \theta^2}} V_z'(\theta),$$

[15] In this section, it is convenient to use the notation a_1 and a_2 for the velocities of the longitudinal and transverse waves respectively.

$$V'_{2x}(\theta) = -2a_2^2\theta\sqrt{a_2^{-2} - \theta^2}\, V_z(\theta),$$

$$V'_{1z}(\theta) = (1 - 2a_2^2\theta^2)\, V'_z(\theta),\ V'_{2z}(\theta) = 2a_2^2\theta^2 V'_z(\theta),$$

$$\Sigma'_{1xz}(\theta) = -\Sigma'_{2xz}(\theta) = -2\mu\theta(1 - 2a_2^2\theta^2)\, V'_z(\theta), \tag{8.8}$$

$$\Sigma'_{1z}(\theta) = -\frac{4\mu a_2^2(\theta^2 - 0.5a_2^{-2})^2}{\sqrt{a_1^{-2} - \theta^2}}\, V'_z(\theta),$$

$$\Sigma'_{2z}(\theta) = -4\mu a_2^2\theta^2\sqrt{a_2^{-2} - \theta^2}\, V'_z(\theta).$$

Here, the prime denotes differentiation with respect to the argument. Equations (8.2) and (8.8) include the expressions for only those components which appear in the boundary conditions. The relation

$$\Sigma'_z(\theta) = \Sigma_{1z}(\theta) + \Sigma_{2z}(\theta) = -\frac{4\mu a_2^2 R(\theta^2)}{\sqrt{a_1^{-2} - \theta^2}}\, V'_z(\theta) \tag{8.9}$$

may be obtained directly from Eqs. (8.8). Here, $R(\theta^2)$ is the Rayleigh function (7.39).

We shall also give the expressions for the accelerations $\dot{v}_z(r, z, t)$ and the rates of stresses $\dot{\sigma}_z(r, z, t)$ for $z = 0$. These can be obtained from (8.2) in the following form:

$$\dot{v}_z(r, 0, t) = \mathrm{Re}\int_{-\pi}^{\pi} V'_z(\theta)\frac{d\varphi}{r\cos\varphi},$$

$$\dot{\sigma}_z(r, 0, t) = \mathrm{Re}\int_{-\pi}^{\pi} \Sigma'_z(\theta)\frac{d\varphi}{r\cos\varphi}. \tag{8.10}$$

The dot indicates differentiation with respect to time.

Using the fact that the function $V_z(\theta)$ is even, we go over to a new argument $\nu = \theta^2$ and introduce the function

$$F(\nu) = V_z(\theta). \tag{8.11}$$

Equations (8.10) can then be written in the form

$$\frac{r}{2}\,\dot{\sigma}_z(r, 0, t) = \mathrm{Re}\int_{l_\nu} \frac{G'(\nu)d\nu}{\sqrt{\nu - \nu_0}},$$

$$\frac{r}{2}\,\dot{v}_z(r, 0, t) = \mathrm{Re}\int_{l_\nu} \frac{F'(\nu)d\nu}{\sqrt{\nu - \nu_0}}. \tag{8.12}$$

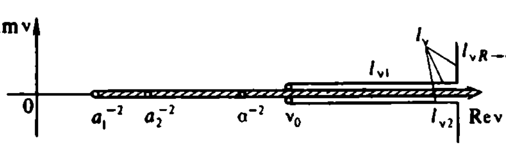

Fig. 47. The l_ν contour.

The notation $G(\nu) = \Sigma_z(\theta)$, $\nu_0 = t^2/r^2$ has been used in this case. The contour l_ν is shown in Fig. 47. The following equality should be noted:

$$G'_\nu = -\frac{4\mu a_2^2}{\sqrt{a_1^{-2} - \nu^2}} R(\nu)F'(\nu). \tag{8.13}$$

In order to uniquely determine the radicals $\sqrt{\nu - \nu_0}$, $\sqrt{a_1^{-2} - \nu}$, and $\sqrt{a_2^{-2} - \nu}$, we make cuts for each of them in the plane ν along the real half-axis from the points ν_0, a_1^{-2}, and a_2^{-2} respectively to infinity. Besides, we require that, for $\nu = 0$, the first radical should be equal to $i\sqrt{\nu_0}$,while the second and third radicals should be positive.

The following expressions may be obtained from (8.2) for the velocities $v_z(r, 0, t)$ and the stresses $\sigma_z(r, 0, t)$:

$$\begin{aligned} v_z(r, 0, t) &= \sqrt{\nu_0}\,\mathrm{Re}\int_{l_\nu} \frac{F(\nu)d\nu}{\nu\sqrt{\nu - \nu_0}}, \\ \sigma_z(r, 0, t) &= \sqrt{\nu_0}\,\mathrm{Re}\int_{l_\nu} \frac{G(\nu)d\nu}{\nu\sqrt{\nu - \nu_0}}. \end{aligned} \tag{8.14}$$

The zero initial conditions will be satisfied if the functions $F(\nu)$ and $G(\nu)$ are regular for Re $\nu < a_1^{-2}$, as well as when the point $\nu = 0$ is not a pole of the integrands in (8.14). Hence, we can assume that

$$F(\nu) = \int_0^\nu F'(\nu)d\nu, \qquad G(\nu) = \int_0^\nu G'(\nu)d\nu. \tag{8.15}$$

The integration in these formulas must be carried out along the contours lying on the same side of the real axis as the point ν. The relations (8.12)-(8.15), obtained in [39], form a system of equations for determining the function $F'(\nu)$.

Further, for the sake of definiteness, we specify the form of the boundary conditions in the following way. We put

$$\begin{aligned} \sigma_z(r, 0, t) &= \sigma_z^0\left(\frac{r}{t}\right) \qquad (0 \leqslant r < vt), \\ v_z(r, 0, t) &= v_z^0\left(\frac{r}{t}\right) \qquad (vt < r < \infty). \end{aligned} \tag{8.16}$$

Going over to the self-modeling variable $\nu_0 = t^2/r^2$, we introduce the notation

$$g(\nu_0) = \frac{r}{2}\,\dot{\sigma}_z^0\left(\frac{r}{t}\right) \qquad (\nu^{-2} < \nu_0 < \infty),$$
$$f(\nu_0) = \frac{r}{2}\,\dot{v}_z^0\left(\frac{r}{t}\right) \qquad (0 < \nu_0 < \nu^{-2}). \tag{8.17}$$

The first of the boundary conditions (8.17) can then be written in the form

$$g(\nu_0) = -4\mu a_2^2 \operatorname{Re}\left\{\int\limits_{l_{\nu 1}} \frac{R(\nu)F'(\nu)d\nu}{\sqrt{a_1^{-2}-\nu}\sqrt{\nu-\nu_0}} + \int\limits_{l_{\nu 2}} \frac{R(\nu)F'(\nu)d\nu}{\sqrt{a_1^{-2}-\nu}\sqrt{\nu-\nu_0}} + \int\limits_{l_{\nu R}} \frac{R(\nu)F'(\nu)d\nu}{\sqrt{a_1^{-2}-\nu}\sqrt{\nu-\nu_0}}\right\}, \tag{8.18}$$

where $l_{\nu 1}$, $l_{\nu 2}$, and $l_{\nu R}$ are the three segments into which the contour l_ν is divided (see Fig. 47).

Suppose that the following estimate is valid for $\nu \to \infty$:

$$F'(\nu) = o(\nu^{-1}). \tag{8.19}$$

In view of this restriction, the stresses at the origin cannot be infinite. The integral in (8.18), taken over the contour $l_{\nu R}$, will vanish in this case. Denoting by $F'_+(\nu)$ and $F'_-(\nu)$ the limiting values of the function $F'(\nu)$ from above and below on the real axis, we get the following equality from (8.18):

$$g(\nu_0) = 4\mu a_2^2 \int\limits_{\nu_0}^{\infty} \frac{R(\nu)[\operatorname{Im}F'_-(\nu) - \operatorname{Im}F'_+(\nu)]}{\sqrt{\nu - a_1^{-2}}\sqrt{\nu-\nu_0}}\, d\nu. \tag{8.20}$$

Here we have taken into account the relation $R_+(\nu) = R_-(\nu) = R(\nu)$. We can transform this equation by treating it as the Abel equation. This gives

$$\frac{4\mu a_2^2 R(\nu)[\operatorname{Im}F'_-(\nu) - \operatorname{Im}F'_+(\nu)]}{\sqrt{\nu - a_2^{-2}}} = -\frac{1}{\pi}\frac{d}{d\nu}\int\limits_{\nu}^{\infty} \frac{g(\nu_0)d\nu_0}{\sqrt{\nu_0-\nu}} \tag{8.21}$$
$$(\nu^{-2} < \nu < \infty,\ \nu^{-2} < \nu_0 < \infty).$$

The improper integral on the right-hand side converges if $g(\nu_0) = o(\nu_0^{-1/2})$. This condition is satisfied when the above-mentioned restriction is imposed on the stresses at the origin.

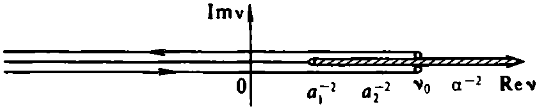

Fig. 48. The transformed l_ν contour.

Let us consider the second condition in (8.17). In this case, we transform the contour l_ν as shown in Fig. 48. Using the analyticity of the function $F'(\nu)$ for $\nu < a_1^{-2}$, it can be shown that the second condition leads to the relation

$$f(\nu_0) = \mathrm{Re}\int_{-\infty}^{\nu_0} \frac{F'_-(\nu)d\nu}{i\sqrt{\nu_0-\nu}} + \mathrm{Re}\int_{\nu_0}^{-\infty} \frac{F'_+(\nu)d\nu}{i\sqrt{\nu_0-\nu}}$$

$$= \int_{a_1^{-2}}^{\nu_0} \frac{\mathrm{Im}F'_-(\nu) - \mathrm{Im}F'_+(\nu)}{\sqrt{\nu_0-\nu}}d\nu. \tag{8.22}$$

Transforming this equation, we get

$$\mathrm{Im}F'_+(\nu) - \mathrm{Im}F'_-(\nu) = -\frac{1}{\pi}\frac{d}{d\nu}\int_{a_1^{-2}}^{\nu} \frac{f(\nu_0)d\nu_0}{\sqrt{\nu-\nu_0}} \tag{8.23}$$

$$(a_1^{-2} < \nu, \quad \nu_0 < v^{-2}).$$

In the interval $-\infty < \nu < a_1^{-2}$, the function $F'(\nu)$ is analytical. Hence, the following equality holds:

$$F'_+(\nu) - F'_-(\nu) = 0 \qquad (-\infty < \nu < a_1^{-2}). \tag{8.24}$$

Taking (8.5) into account, we get the equality

$$\overline{F'(\nu)} = F'(\bar{\nu}). \tag{8.25}$$

It follows from the above that in view of (8.17), (8.19), and (8.24), we can go over to the Riemann problem for the analytical function $F(\nu)$:

$$F^+(\nu) - F^-(\nu) = g(\nu), \qquad (-\infty < \nu < a_1^{-2}) \tag{8.26}$$

$$g(\nu) = \begin{cases} 0 & \\ -\dfrac{i}{\pi}\dfrac{d}{d\nu}\displaystyle\int_{a_1^{-2}}^{\nu} \frac{f(\nu_0)d\nu_0}{\sqrt{\nu_0-\nu}} & (a_1^{-2} < \nu < v^{-2}), \\ \dfrac{i\sqrt{\nu - a_1^{-2}}}{4\pi\mu a_2^2 R(\nu)}\dfrac{d}{d\nu}\displaystyle\int_{\nu}^{\infty} \frac{g(\nu_0)d\nu_0}{\sqrt{\nu_0-\nu}} & (v^{-2} < \nu < \infty). \end{cases}$$

We shall proceed from a somewhat more general formulation of the Riemann problem for the case of discontinuous coefficients than the formulation given in Sec. 1, Ch. 1, Vol. 1, by assuming singularities of the type of δ-functions at the points a_1^{-2}, v^{-2}. It should be noted that in view of the condition (8.19), there can be no singularity at infinity. It can also be shown that the existence of a pole at the point a_1^{-2} would lead to infinite stresses on the longitudinal wavefront, which we shall also exclude. Hence, the general solution of the Riemann problem (8.26) may be represented in the following form (A_j are constants):

$$F'(\nu) = \frac{1}{2\pi i} \int\limits_{a_1^{-2}}^{\infty} \frac{\Phi(\tau)d\tau}{\tau - \nu} + \sum_{j=1}^{m} \frac{A_j}{(\nu - v^{-2})_j}. \tag{8.27}$$

This representation has been obtained by eliminating terms due to a possible singularity at the point v^{-2}.

The first integral in (8.27) does not ensure that the condition (8.19) is satisfied, since it tends to zero as $1/\nu$ for $\nu \to \infty$. In order to ensure that this condition is satisfied, the constant A_1 must be chosen in an appropriate manner. The order m of the pole as well as the remaining constants A_j ($j \neq 1$) are found from the additional condition on the line where the boundary conditions change.

The case when the velocities $v_z(r, 0, t)$ are given on the part $r < vt$ of the boundary and the stresses $\sigma_z(r, 0, t)$ are given on the part $vt < r < \infty$ is considered in a similar manner and leads to a problem of the type (8.26) for the function $G'(\nu)$. Such situations are encountered while considering the dynamic problems of indentation by rigid conical punches in an elastic half-space.

As an example, let us consider a problem of the growth of a crack under a constant load. In this case, $g(\nu_0) = f(\nu_0) = 0$, and only the sum $\sum\limits_{j=1}^{m} \frac{A_j}{(\nu - v^{-2})^j}$ is retained in the expression (8.27). Let us consider the arguments which enable us to specify this sum. We require that the stress σ_z should increase as $\delta^{-1/2}$ ($\delta = |r - vt| \to 0$). The derivative $\dot{\sigma}_z$ should then increase as $\delta^{-3/2}$, which is possible only for $n = 2$. Thus, we get

$$F'(\nu) = \frac{A}{(\nu - v^{-2})^2}. \tag{8.28}$$

Consequently,

$$F(\nu) = \frac{v^2 \nu A}{v^{-2} - \nu}. \tag{8.29}$$

By transforming the contour of integration, the expression for the function $G(\nu)$ can be represented in a modified form as follows:

$$G(\nu) = \int_0^{\nu} G'(\nu)\,d\nu = \int_0^{-\infty} G'(\nu)\,d\nu + \int_{\infty\nu}^{\nu} G'(\nu)\,d\nu$$

$$= -4\mu a_2^2 \left[\int_0^{\infty} \frac{R(\nu)}{\sqrt{\nu^{-2} - \nu}} F'(\nu)\,d\nu + \int_{\infty\nu}^{\nu} \frac{R(\nu)F'(\nu)\,d\nu}{\sqrt{\nu^{-2} - \nu}} \right] = M + G_1(\nu), \quad (8.30)$$

where M is a constant (first integral).

It follows from (8.14) that v_z actually vanishes for $r = vt$. From the expression for $G_1(\nu)$ in (8.30), we find that this function changes sign upon passing through the cut form ν^{-2} to ∞. Hence, the integral in the second formula in (8.14) vanishes. Thus, we get an expression for the stresses σ_z in the form

$$\sigma_z = \int_{l_\nu} \frac{d\nu}{\nu\sqrt{\nu - \nu_0}} M\sqrt{\nu_0} = -2M \qquad (z = 0, r < vt). \quad (8.31)$$

On the other hand, $\sigma_z = \sigma_z^0$ (for $z = 0, r < vt$), which leads to an explicit expression for the constant

$$\sigma_z^0 = 2A \int_0^{\infty} \frac{\left(\nu + \frac{1}{2}a_2^{-2}\right) - \nu\sqrt{a_1^{-2} + \nu}\sqrt{a_2^{-2} + \nu}}{(\nu^{-2} + \nu)^2\sqrt{a_1^2 + \nu}}\,d\nu. \quad (8.23)$$

Similarly, the expression for v_z when $z = 0$ and $r < vt$ can be written as follows:

$$v_z = -\frac{v^3 A t}{2\pi\sqrt{v^2t^2 - r^2}}. \quad (8.33)$$

Integrating this expression, we arrive at the following equation for the edges of the cut:

$$u_z = -2\pi A v\sqrt{v^2t^2 - r^2}. \quad (8.34)$$

Chapter Six
Integral Representations and Integral Transformations

Section 1
Problems in the Theory of Elasticity for a Strip and a Layer

Let us consider the plane problem in the theory of elasticity for the case of a strip, i.e. for the interval $-\infty < x < \infty$, $|y| \leqslant a$ [40]. It is required to determine the stresses σ_x, σ_y, τ_{xy} in this region, satisfying the equilibrium equations (4.4), Ch. 3, Vol. 1:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0.$$

Besides, these stresses should satisfy the Beltrami-Michell equations (4.7), Ch. 3, Vol. 1:

$$\Delta(\sigma_x + \sigma_y) = 0$$

and should assume the given values at the boundary:

$$\sigma_y(x, y)|_{y = \pm a} = f(x), \qquad \tau_{xy}(x, y)|_{y = \pm a} = \pm g(x). \tag{1.1}$$

We introduce the condition

$$\int_{-\infty}^{\infty} g(x)dx = 0, \tag{1.2}$$

which is equivalent to the condition that the stresses at infinity should vanish. The necessity of satisfying the condition (1.2) is dictated by mathematical considerations which we shall be considering later. Here, for the sake of simplicity, the boundary conditions have been chosen in such a way that the x-axis is the axis of symmetry.

Instead of the stress components $\sigma_x(x, y)$, $\sigma_y(x, y)$, and $\tau_{xy}(x, y)$ we shall be considering their Fourier transforms $\bar{\sigma}_x(\lambda, y)$, $\bar{\sigma}_y(\lambda, y)$, and $\bar{\tau}_{xy}(\lambda, y)$:

$$\bar{\sigma}_x(\lambda, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x(x, y)e^{i\lambda x}dx,$$

$$\bar{\sigma}_y(\lambda, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_y(x, y)e^{i\lambda x}dx,$$

$$\bar{\tau}_{xy}(\lambda, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau_{xy}(x, y) e^{i\lambda x} dx.$$

We also take the transforms $\bar{f}(\lambda)$ and $\bar{g}(\lambda)$ of the functions $f(x)$ and $g(x)$:

$$\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx, \qquad \bar{g}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{i\lambda x} dx.$$

Naturally, it is assumed (for the functions $f(x)$ and $g(x)$, it is essential) that all the Fourier transforms introduced above do exist.

Let us now form equations for the transforms, similar to Eqs. (4.4) and (4.7). Acting by the Fourier operator on all the terms of, say, the first equation in (4.4), we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \sigma_x(x, y)}{\partial x} e^{i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{i\lambda x} \sigma_x \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \sigma_x(x, y)(i\lambda) e^{i\lambda x} dx \right] = -i\lambda \bar{\sigma}_x(\lambda, y).$$

The second term can be easily found, since in this case, differentiation with respect to x means differentiation with respect to the parameter:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \tau_{xy}(x, y)}{\partial y} e^{i\lambda x} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial y} \int_{-\infty}^{\infty} \tau_{xy}(x, y) e^{i\lambda x} dx = \bar{\tau}'_{xy}(\lambda, y),$$

where the prime indicates differentiation with respect to y.

Thus, the first equation in (4.4) can be transformed as follows:

$$-i\lambda \bar{\sigma}_x + \bar{\tau}'_{xy}(\lambda, y) = 0. \tag{1.3}$$

Similarly (i.e. assuming that the transform of the derivative with respect to y is the derivative of the transform with respect to y, while the transform of the derivative with respect to x is the transform of the original function, multiplied by $-i\lambda$), we get the following equations corresponding to the second equation in (4.4) and Eq. (4.7):

$$-i\lambda \bar{\tau}_{xy} + \bar{\sigma}'_y = 0, \tag{1.4}$$

$$(\bar{\sigma}_x + \bar{\sigma}_y)'' - \lambda^2(\bar{\sigma}_x + \bar{\sigma}_y) = 0. \tag{1.5}$$

Thus, the solution of a system of equations in partial derivatives has been reduced to the solution of a system of ordinary differential equations (1.3)-(1.5) (the variable

y plays the role of a parameter), which must be solved under the following boundary conditions:

$$\bar{\sigma}_y(\lambda, \pm a) = \bar{f}(\lambda), \qquad \bar{\tau}_{xy}(\lambda, \pm a) = \bar{g}(\lambda). \tag{1.6}$$

Eliminating the functions σ_x and $\bar{\tau}_{xy}$ from the system (1.3)-(1.5), we get a fourth-order equation:

$$\bar{\sigma}_y^{\mathrm{IV}} - 2\lambda^2\bar{\sigma}_y^{\mathrm{II}} + \lambda^4\bar{\sigma}_y = 0. \tag{1.7}$$

Instead of the second equation in (1.6), we now have

$$\bar{\sigma}_y'(\lambda, \pm a) = i\lambda\bar{g}(\lambda). \tag{1.8}$$

The general solution of the system (1.3)-(1.5) has the form

$$\bar{\sigma}(\lambda, y) = A(\lambda)\cosh\lambda y + B(\lambda)\lambda y\sinh\lambda y, \tag{1.9}$$

where $A(\lambda)$ and $B(\lambda)$ are arbitrary functions obtained from the conditions (1.6) and (1.8). Finally, we get

$$\left.\begin{aligned} A &= 2\frac{\bar{f}(\mu\cosh\mu + \sinh\mu) - \bar{g}i\mu\sinh\mu}{\sinh 2\mu + 2\mu}, \\ B &= 2 - \frac{\bar{f}\sinh\mu + \bar{g}i\cosh\mu}{\sinh 2\mu + 2\mu} \end{aligned}\right\} \quad (\mu = \lambda a). \tag{1.10}$$

The expressions for the transforms $\bar{\sigma}_x(\lambda, y)$ and $\bar{\tau}_{xy}(\lambda, y)$ are as follows:

$$\bar{\sigma}_x = -A\cosh\lambda y + B(\lambda y\sinh\lambda y + 2\cosh\lambda y), \tag{1.11}$$

$$\bar{\tau}_{xy} = A\sinh\lambda y + B(\lambda y\cosh\lambda y + \sinh\lambda y). \tag{1.12}$$

The required stresses can be found by carrying out inverse transformations.

One important point is worth noticing. For $\lambda = 0$, the coefficient B becomes infinite and the integrals for the originals diverge. The necessary condition for the boundedness of the coefficient B is that the equation $\bar{g}(0) = 0$ be satisfied. This follows from the condition (1.2):

$$\bar{g}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)dx.$$

For an actual calculation of the originals from the transforms (1.10)-(1.12), it is necessary to know the zeros of the function $\sinh 2\mu + 2\mu$, appearing in the denominator of the expressions (1.10) for $A(\lambda)$ and $B(\lambda)$. This is an essential requirement for calculating the integrals with the help of the residue theory.

Using the Fourier transform for a function of two variables (Eq. (4.17), Ch. 1, Vol. 1), we can extend the method described above to the problems in the theory of elasticity for a layer [41]. Suppose that an elastic medium occupies a region $-\infty < x, y < \infty$, $|z| \leqslant h$. For the sake of simplicity, we shall assume that the plane $z = 0$ is the plane of symmetry for the solution. The boundary conditions on the upper and lower sides of the layer may be written as follows:

$$\sigma_z(x,y,z)|_{z=\pm h} = f(x,y), \tag{1.13}$$

$$\begin{aligned} \tau_{xz}(x,y,z)|_{z=\pm h} &= \pm g_1(x,y), \\ \tau_{yz}(x,y,z)|_{z=\pm h} &= g_2(x,y). \end{aligned} \tag{1.14}$$

In order that this method be applicable in the present case, it is necessary that the following equalities be satisfied:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(x,y)dxdy = 0. \tag{1.15}$$

We proceed from the Beltrami-Michell equations (4.11), (4.16), (4.17), Ch. 2, Vol. 1:

$$\Delta\sigma_z + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial z^2} = 0,$$

$$\Delta\tau_{xz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x\partial z} = 0,$$

$$\Delta\tau_{yz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\partial z} = 0,$$

as well as from Eq. (4.6), Ch. 2, Vol. 1:

$$\Delta\Theta = 0.$$

Acting by the Fourier operator (with respect to the variables x and y), we get the following ordinary differential equations for the transforms:

$$\begin{aligned} \frac{d^2\bar{\sigma}_z}{dz^2} - \gamma^2\bar{\sigma}_z &= -\frac{1}{1+\sigma}\frac{d^2\bar{\Theta}}{dz^2}, \\ \frac{d^2\bar{\tau}_{xz}}{dz^2} - \gamma^2\bar{\tau}_{xz} &= -\frac{i\lambda_x}{1+\sigma}\frac{d\bar{\Theta}}{dz}, \\ \frac{d^2\bar{\tau}_{yz}}{dz^2} - \gamma^2\bar{\tau}_{yz} &= -\frac{i\lambda_y}{1+\sigma}\frac{d\bar{\Theta}}{dz}, \\ \frac{d^2\bar{\Theta}}{dz^2} - \gamma^2\bar{\Theta} &= 0 \qquad (\lambda = (\lambda_x, \lambda_y), \gamma = |\lambda|). \end{aligned} \tag{1.16}$$

The general solution of this system, which is even in z, is of the form

$$\begin{aligned} \bar{\sigma}_z &= 2A_z\cosh\gamma z - \frac{A\gamma z}{1+\sigma}\sinh\gamma z, \\ \bar{\tau}_{xz} &= -2A_x\sinh\gamma z - \frac{i\lambda_x zA}{1+\sigma}\cosh\gamma z, \\ \bar{\tau}_{yz} &= -2A_y\sinh\gamma z - \frac{i\lambda_y zA}{1+\sigma}\cosh\gamma z, \end{aligned} \tag{1.17}$$

where A, A_x, A_y, and A_z are functions of the parameters λ_x and λ_y, related to each other as per equation

$$A = \frac{2(1+\sigma)}{\gamma}(A_z\gamma - i\lambda_x A_x - i\lambda_y A_y). \tag{1.18}$$

Let $\bar{f}$, $\bar{g}_1$, and $\bar{g}_2$ be the Fourier transforms of the boundary conditions $f(x, y)$, $g_1(x, y)$, and $g_2(x, y)$ respectively. Then, equating their values to the right-hand sides of Eqs. (1.17), we get a system of equations for determining the functions A, A_x, A_y, and A_z. The solution of this system is given by

$$\begin{aligned}
A &= \frac{1+\sigma}{\Delta_1}\left[\bar{f}\sinh\gamma h + \frac{1}{\gamma}(i\lambda_x\bar{g}_1 + i\lambda_y\bar{g}_2)\cosh\gamma h\right],\\
A_x &= -\frac{1}{2\sinh\gamma h}\bar{g}_1 - \frac{i\lambda_x hA}{2(1+\sigma)}\coth\gamma h,\\
A_y &= -\frac{1}{2\sinh\gamma h}\bar{g}_2 - \frac{i\lambda_y hA}{2(1+\sigma)}\coth\gamma h,\\
A_z &= \frac{1}{2\cosh\gamma h} + \frac{\gamma hA}{2(1+\sigma)}\tanh\gamma h,\\
\Delta_1 &= \frac{1}{2}(\sinh 2\gamma h + 2\gamma h).
\end{aligned} \tag{1.19}$$

It should be noted that a restoration of the stresses in accordance with (1.17) and (1.19), which can be reduced to a solution of fourth-order integrals, is possible only when the condition (1.15) is satisfied (otherwise, the integrals are found to be divergent).

It follows from symmetry considerations that the formulas (1.17) and (1.19) yield the solution of the problem for a layer of thickness h, on whose bottom $z = 0$ the shearing stresses and the normal component of displacements vanish (in other words, the elastic layer rests on a rigid substrate). Let us consider the case when the boundary values of shearing stresses vanish at $z = h$ [42]. The expression for the displacement component w can be written as follows:

$$w = \frac{1}{2\pi G}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{Q(\lambda_x, \lambda_y)}{\gamma(2\gamma h + \sinh 2\gamma h)}\Big[\gamma h\cosh\gamma h\sinh\gamma z$$

$$- \gamma z\cosh\gamma z\sinh\gamma h + 2\left(1 - \frac{2}{\sigma}\right)\sinh\gamma z\sinh\gamma h\Big]e^{i(\lambda_x x + \lambda_y y)}d\lambda_x d\lambda_y, \tag{1.20}$$

where

$$Q(\lambda_x, \lambda_y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\sigma_z]_{z=h}e^{-i(\lambda_x x + \lambda_y y)}dx\,dy.$$

Let us apply formulas (1.20) to the solution of the problem, when a smooth rigid punch is acting on the surface element Ω of the upper part of the layer, while outside Ω all stresses vanish. Suppose that $f(x, y)$ is the required contact pressure. Then (1.20) gives the following integral equation for $f(x, y)$:

$$\iint_{\Omega} f(x', y')K(x - x', y - y', h)dx'dy' = 2\pi^2 \frac{G}{1 - \sigma} F(x, y),$$

where $F(x, y)$ is the punch surface (taking into account the constants determining the displacement of the punch as a rigid entity), and

$$K(x - x', y - y', h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sinh^2 \gamma h}{\gamma(2\gamma h + \sinh 2\gamma h)} e^{i[\lambda_x(x - x') + \lambda_y(y - y')]} d\lambda_x d\lambda_y.$$

In [42] the authors give a method of solution effective for $h \geqslant d/\sqrt{2}$, where d is the diameter of the domain Ω.

Let us now consider the mixed problem for a strip also on the basis of the Fourier transformation [43]. Before specifying the boundary conditions, let us formulate the original boundary value problem whose approximation is just the problem for a strip, since the meaning of the boundary conditions then becomes clear.

Suppose that we have in an infinite plane a number of identical straight cuts located one above the other and having their sides free of load. Let a shearing stress τ_0 be applied at infinity (in the coordinate system with the axes directed along the cuts). Let $2c$ be the length of the cuts and $2b$ be the distance between them. It follows from symmetry considerations that we can go over to the problem for a strip $|y| \leqslant b$ under the conditions of antisymmetric loading:

$$\begin{aligned} &\sigma_y(x, y)|_{y = \pm b} = 0 && (-\infty < x < \infty), \\ &u(x, y)|_{y = \pm b} = 0 && (x < 0, x > 2c), \\ &\tau_{xy}(x, y)|_{y = \pm b} = \tau_0 && (0 \leqslant x \leqslant 2c). \end{aligned} \tag{1.21}$$

Here, the origin of coordinates is chosen in such a way that it lies in front of the left end of the cuts. The following equations are satisfied for the shearing stress on the extension of the cuts:

$$\int_{-\infty}^{0} \tau(x, b)dx = \int_{2c}^{\infty} \tau(x, b)dx = \tau_0 c. \tag{1.22}$$

We shall assume that the cuts are close to one another ($c \gg b$). Then the self-balancing loads applied to the strip in the region $x > c$ have a negligible effect on the stress distribution in the vicinity of the left end of a cut ($x = 0$). Hence, while investigating the stresses in this region, we go over, for the sake of simplicity, to another problem, i.e. the problem for an infinite strip under the conditions

$$\begin{aligned} &\sigma_y(x, y)|_{y = \pm b} = 0 && (-\infty < x < \infty), \\ &\tau_{xy}(x, y)|_{y = \pm b} = \tau_0 && (x > 0), \\ &u(x, y)|_{y = \pm b} = 0 && (x < 0). \end{aligned} \tag{1.23}$$

In this case, we assume that an infinite moment is applied at infinity, thus making the resultant moment equal to $2cb\tau_0$ at the cross section $x = 0$.

Note that the following equation results from the condition $\sigma_y = 0$ imposed on the edges of the strip:

$$\left.\frac{\partial u}{\partial x}\right|_{y=\pm b} = \frac{\sigma_x}{E}. \tag{1.24}$$

Unlike the approach followed above, we use Airy's function $U(x, y)$ (Sec. 4, Ch. 3, Vol. 1).

Let us now assume that there exists a constant $\beta > 0$, such that the function $\exp(\alpha x)U(x, y)$ is absolutely integrable with respect to x between $-\infty$ and ∞ for $0 < \alpha < \beta$, the function $\exp(-\alpha x)\,u(x)(u(x) = u(x, \pm b))$ is absolutely integrable between 0 and ∞ for $\alpha > \beta$ and the function $\exp(-\alpha x)\,t(x)$ is absolutely integrable between $-\infty$ and 0 for $\alpha < \beta$ $(t(x) = \tau_{xy}(x, \pm b))$.

We go over to the Fourier transforms

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(x,y)e^{i\lambda x}dx = F(\lambda,y), \tag{1.25}$$

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} u(x)e^{i\lambda x}dx = U_+(\lambda), \tag{1.26}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} t(x)e^{i\lambda x}dx = T_-(\lambda), \tag{1.27}$$

$$\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \tau_0 e^{i\lambda x}dx = \frac{i\tau_0}{\sqrt{2\pi}}\frac{1}{\lambda}\,. \tag{1.28}$$

From the condition introduced above, it follows that the function $F(\lambda, y)$ is regular on the interval $0 < \operatorname{Im}\lambda < \beta$, $U_+(\lambda)$ on the half-plane $\operatorname{Im}\lambda > 0$, and $T_-(\lambda)$ on the half-plane $\operatorname{Im}\lambda < \beta$. The function $\dfrac{i\tau_0}{\sqrt{2\pi}}\dfrac{1}{\lambda}$ is regular on the half-plane $\operatorname{Im}\lambda > 0$. We obtain the following equation for the Fourier transform and for the Airy function:

$$\frac{d^4F}{dy^4} - 2\lambda^2\frac{d^2F}{dy^2} + \lambda^4F = 0. \tag{1.29}$$

The transforms of the boundary conditions are connected through the relations (for $y = \pm b$), obtained with the help of Eq. (1.24)

$$-\lambda^2F = 0, \quad i\lambda\frac{dF}{dy} = T_-(\lambda) - \frac{i\tau_0}{\sqrt{2\pi}}\frac{1}{\lambda}, \qquad \frac{1}{E}\frac{d^2F}{dy^2} = -i\lambda U_+(\lambda). \tag{1.30}$$

The solution of Eq. (1.29), satisfying the boundary conditions (1.30), is given by the expression

$$E\lambda U_+(\lambda) = \frac{2\sinh^2\lambda b}{\sinh\lambda b\cosh\lambda b - \lambda b}\left[T_-(\lambda) - \frac{i\tau_0}{\sqrt{2\pi}}\frac{1}{\lambda}\right]. \tag{1.31}$$

This is a factorization problem (see Sec. 4, Ch. 1, Vol. 1) for the interval $0 < \operatorname{Im}\lambda < \beta$. Yet another condition

$$T_-(0) = \frac{\tau_0 c}{\sqrt{2\pi}} \tag{1.32}$$

follows from (1.22).

In order to solve Eq. (1.31), we introduce the auxiliary function $H(\lambda b)$:

$$\frac{\sinh\lambda b\cosh\lambda b - \lambda b}{\sinh^2\lambda b} = \frac{\lambda b}{\sqrt{\lambda^2 b^2 + 9/4}}H(\lambda b), \tag{1.33}$$

where the positive branch of the square root should be taken on the real axis. The function $H(\lambda b)$ is regular and does not vanish in the interval $-3/2 < \operatorname{Im}\lambda b < 3/2$ containing the real axis. Moreover, $H(0) = 1$ and $H(\lambda b) \to 1 + O[(\lambda b)^{-2}]$ for $|\lambda b| \to \infty$ in the regularity interval. Consequently, the function

$$\chi(\lambda b) = \ln H(\lambda b) \tag{1.34}$$

(the real branch of the logarithm is taken on the real axis) is also regular in this interval. We form the functions

$$\chi_+(\lambda b) = \frac{1}{2\pi i}\int_{-i\gamma b - \infty}^{-i\gamma b + \infty}\frac{\chi(\omega)}{\omega - \lambda b}\,d\omega$$

$$(0 < \gamma b < 3/2), \tag{1.35}$$

$$\chi_-(\lambda b) = \frac{1}{2\pi i}\int_{i\gamma b - \infty}^{i\gamma b + \infty}\frac{\chi(\omega)}{\omega - \lambda b}\,d\omega$$

which are regular on the half-planes $\operatorname{Im}\lambda > -\gamma$ and $\operatorname{Im}\lambda < \gamma$ respectively, and tend to zero for $|\lambda b| \to \infty$. Obviously, $\chi(\lambda b) = \chi_+(\lambda b) - \chi_-(\lambda b)$. We form the functions

$$H_+(\lambda b) = e^{\chi_+(\lambda b)}, \quad H_-(\lambda b) = e^{\chi_-(\lambda b)}. \tag{1.36}$$

Thus, the factorization problem can be represented in the form

$$\frac{E\lambda b}{\sqrt{\lambda b + (3/2)i}}H_+(\lambda b)U_+(\lambda) = 2\sqrt{\lambda b - (3/2)i}\,H_-(\lambda b)\left[\lambda T_-(\lambda) - \frac{i\tau_0}{\sqrt{2\pi}}\right], \tag{1.37}$$

where the principal values of the square roots are taken. The left-hand side of (1.37) is regular on the upper half-plane $\operatorname{Im}\lambda > 0$, while the right-hand side is regular on

the lower half-plane $\operatorname{Im}\lambda < \gamma$. Consequently, they represent a single analytical function on the entire plane. Since $H_-(\lambda b)$ tends to unity for $|\lambda b| \to \infty$ on the lower half-plane, while $T_-(\lambda)$ (as the Fourier transform) must tend to zero for $|\lambda| \to \infty$ on the lower half-plane, the single function must be a linear function on the plane λ:

$$C\lambda + D. \tag{1.38}$$

We then get the following representations for $T_-(\lambda)$ and $U_+(\lambda)$ from (1.37):

$$\begin{aligned} T_-(\lambda) &= \frac{i\tau_0}{\sqrt{2\pi}}\frac{1}{\lambda} + \frac{C\lambda + D}{2\lambda\sqrt{\lambda b - (3/2)i}H_-(\lambda b)}, \\ U_+(\lambda) &= \frac{(C\lambda + D)\sqrt{\lambda b + (3/2)i}}{E\lambda^3 b H_+(\lambda b)}. \end{aligned} \tag{1.39}$$

From the condition (1.32) of regularity of $T_-(\lambda)$ for $\lambda = 0$, we get the equation

$$D = -\frac{2i\tau_0}{\sqrt{2\pi}}\sqrt{-(3/2)i}H_-(0). \tag{1.40}$$

Since the value of $T_-(0)$ is known, we get the value of the constant C:

$$C = \frac{2\tau_0}{\sqrt{2\pi}}\sqrt{-(3/2)i}H_-(0)\left\{c + b\left[\frac{1}{3} - i\chi'(0)\right]\right\}, \tag{1.41}$$

where $\chi'(\lambda b)$ is the derivative with respect to λb.

With this discussion, we end the construction of the transforms.

In order to go over to physical quantities, we must carry out an inverse transformation. For example, the displacement is obtained in the form of the integral

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{i\gamma - \infty}^{i\gamma + \infty} U_+(\lambda)e^{-i\lambda x}dx, \tag{1.42}$$

where $0 < \gamma b < 3/2$. Closing the integration contour in the upper plane, we can easily find that the condition $u(x) = 0$ is satisfied for $x < 0$.

Section 2
Analysis of a Wedge. Carother's Paradox

Suppose that an elastic medium fills a region in the form of a wedge. We use the polar system of coordinates, in which the Airy function U (Eqs. (4.20), Ch. 3, Vol. 1) must satisfy the equation .

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 U = 0. \tag{2.1}$$

The components of the stress tensor are related to the Airy function through the

following equalities:

$$\sigma_r = \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2},$$
$$\sigma_\theta = \frac{\partial^2 U}{\partial r^2}, \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial \theta}\right). \tag{2.2}$$

Suppose that the wedge has an angle of 2α ($-\alpha \leqslant \theta \leqslant \alpha$). We shall assume that the stresses

$$\sigma_\theta = f_1(r), \qquad \tau_{r\theta} = g_1(r) \qquad (\theta = \alpha),$$
$$\sigma_\theta = f_2(r), \qquad \tau_{r\theta} = g_2(r) \qquad (\theta = -\alpha) \tag{2.3}$$

are acting on the sides of the wedge. The restrictions which must be imposed on the boundary condition will be discussed later.

In order to solve the above boundary value problem, we use Mellin's transform (4.31), Ch. 1, Vol. 1. In other words, we go over from the required function $U(r, \theta)$ to its transform $\bar{U}(p, \theta)$:

$$\bar{U}(p, \theta) = \int_0^\infty r^{p-1} U(r, \theta) dr. \tag{2.4}$$

Further, we assume the function U to be such that the expressions of the type

$$r^{p+n}\frac{\partial^n U}{\partial r^n} \qquad (n = 0, 1, 2, 3),$$
$$r^p \frac{\partial^n U}{\partial \theta^n} \qquad (n = 1, 2), \qquad r^{p+1}\frac{\partial^3 U}{\partial r \partial \theta^2}$$

tend to zero as r tends to infinity. Integrating (2.4) by parts, we get

$$\int_0^\infty r\frac{\partial U}{\partial r} r^{p-1} dr = -p\bar{U},$$
$$\int_0^\infty r^2\frac{\partial^2 U}{\partial r^2} r^{p-1} dr = p(p+1)\bar{U},$$
$$\int_0^\infty r^3\frac{\partial^3 U}{\partial r^3} r^{p-1} dr = -p(p+1)(p+2)\bar{U}, \tag{2.5}$$
$$\int_0^\infty r^4\frac{\partial^4 U}{\partial r^4} r^{p-1} dr = p(p+1)(p+2)(p+3)\bar{U}.$$

Differentiating first with respect to the angle θ, which is considered as a parameter, we get

$$\int_0^\infty r\frac{\partial^{n+1}U}{\partial r\partial\theta^n}r^{p-1}dr = -p\frac{d^n\bar{U}}{d\theta^n} \qquad (n = 0, 1, 2),$$
$$\int_0^\infty r^2\frac{\partial^{n+2}U}{\partial r^2\partial\theta^n}r^{p-1}dr = p(p+1)\frac{d^n\bar{U}}{d\theta^n} \qquad (n = 0, 2). \tag{2.6}$$

We perform Mellin's transformation to all the terms appearing in Eq. (2.1). Then, with the help of the formulas (2.5) and (2.6) obtained above, we arrive at an ordinary differential equation

$$\left(\frac{d^2}{d\theta^2}+p^2\right)\left[\frac{d^2}{d\theta^2}+(p+2)^2\right]\bar{U} = 0. \tag{2.7}$$

Formulas (2.2) can be used to obtain the expressions for the transforms of stresses. However, in order to directly employ the apparatus of Mellin's transforms, it is better to use the formulas for the transforms of the products of stresses and the squares of distances, i.e. $r^2\sigma_r$, $r^2\sigma_\theta$, and $r^2\tau_{r\theta}$. The corresponding transforms are still denoted by $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, and $\bar{\tau}_{r\theta}$. The following equalities can then be obtained from the formulas (2.5) and (2.6):

$$\bar{\sigma}_r = \left(\frac{d^2}{d\theta^2}-p\right)\bar{U}(p,\theta), \tag{2.8}$$

$$\bar{\sigma}_\theta = p(p+1)\bar{U}(p,\theta), \tag{2.9}$$

$$\bar{\tau}_{r\theta} = (p+1)\frac{d\bar{U}(p,\theta)}{d\theta}. \tag{2.10}$$

The general solution of Eq. (2.7) has the form

$$U = A(p)\sin(p\theta) + B(p)\cos(p\theta) + C(p)\sin[(p+2)\theta] + D(p)\cos[(p+2)\theta]. \tag{2.11}$$

The functions $A(p)$, $B(p)$, $C(p)$, and $D(p)$ are found from the boundary conditions for Eq. (2.7), i.e. from the expressions (2.9) and (2.10) by substituting the values $\theta = \pm\alpha$ in them. In this case, these functions can be directly expressed in terms of integrals of the real boundary values (2.3).

Making the reverse transformation to the original (with the help of the integral (4.32), Ch. 1, Vol. 1), we get a formal solution of the problem in the form of double integrals.

We shall now confine ourselves to the case of an antisymmetric load, assuming further that the shearing stresses on the sides vanish [44]. As regards the normal

stresses, we shall assume that $f_1(r) = -f_2(r)$, and that $f(r) = 0$ if r is larger than a certain value a (for the sake of convenience, we shall henceforth omit the subscript of the function f).

Then, the function $\bar{g}(p)$ will be equal to zero, while the function $\bar{f}(p)$ can be represented in the form of the integral[1]

$$\bar{f}(p) = \int_0^a f(r) r^{p+1} dr. \tag{2.12}$$

We shall now require additionally that the resultant of the forces applied to either side vanish. Then, we get the following equation from (2.12), which is applicable to the transform $\bar{f}(p)$:

$$\bar{f}(-1) = 0. \tag{2.13}$$

Without specifying the form of the load for the time being, we simply introduce its integral characteristic—the total moment M applied to both sides. This leads to yet another condition for the transform $\bar{f}(p)$:

$$\bar{f}(0) = \frac{M}{2}. \tag{2.14}$$

In view of antisymmetry, the general solution of Eq. (2.11) can be simplified to the following form:

$$\bar{U} = A(p)\sin(p\theta) + C(p)\sin[(p+2)\theta]. \tag{2.15}$$

The expressions for the functions $A(p)$ and $C(p)$ can be easily obtained:

$$A = -\frac{\bar{f}(p)(p+2)\cos[(p+2)\alpha]}{p(p+1)G(p,\alpha)}, \quad C = \frac{\bar{f}(p)\cos p\alpha}{(p+1)G(p,\alpha)}, \tag{2.16}$$

where

$$G(p,\alpha) = (p+1)\sin 2\alpha - \sin[2(p+1)\alpha]. \tag{2.17}$$

Similarly, the expressions for the transforms $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, and $\bar{\tau}_{r\theta}$ are given by the following equations:

$$\begin{aligned} \bar{\sigma}_r &= -Ap(p+1)\sin p\theta - C(p+1)(p+4)\sin[(p+2)\theta], \\ \bar{\sigma}_\theta &= Ap(p+1)\sin p\theta + Cp(p+1)\sin[(p+2)\theta], \\ \bar{\tau}_{r\theta} &= Ap(p+1)\cos p\theta + C(p+1)(p+2)\cos[(p+2)\theta]. \end{aligned} \tag{2.18}$$

Let us now recall the expressions obtained for each stress component, denoted by σ in order to unify the notation:

$$\sigma = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{\sigma} r^{p-2} dp. \tag{2.19}$$

[1] It should be recalled that we are taking the transform of the product of stresses and r^2, and not of stresses alone.

Here, the constant c is chosen from the condition of the existence of the integrals. It is convenient to put $c = -1$, since Eq. (2.16) implies the regularity of the integrand on the line $(c - i\infty, c + i\infty)$. For the sake of simplicity of analysis, we shall confine ourselves to the solution which is obtained for large values of r. For this purpose, we construct a closed integration contour, supplementing the straight line $c = -1$ by an arc on the right side. In order to apply the residue theory for calculating the integrals, it is necessary to investigate the zeros of the function $G(p, \alpha)$ from Eq. (2.17), situated in the vicinity of the line $c = -1$ on the right side, as a function of the angle α. It should be noted that there are no complex zeros of the function $G(p, \alpha)$ on the line $\text{Re}\,p = -1$.

The expression for the derivative of the function $G(p, \alpha)$ can be written in the following form:

$$\frac{\partial G}{\partial p} = \sin 2\alpha - 2\cos[2(p + 1)\alpha] = 0. \tag{2.20}$$

The values of p, which simultaneously satisfy the equations $G(p, \alpha) = 0$ and $\partial G(p, \alpha)/\partial p = 0$ will be second-order zeros of the function $G(p, \alpha)$.

Let us now consider the point $p = 0$. It will be always a zero; moreover, if (taking (2.20) into account) the angle α satisfies the condition

$$\sin 2\alpha - 2\cos 2\alpha = 0, \tag{2.21}$$

the zero will be multiple. This angle is called critical and is denoted by α^*. It has a value $\alpha^* = 0.715\pi$.

We shall consider three different cases. In the first case, $\alpha < \alpha^*$. Then the function $G(p, \alpha)$ does not have any zero in the interval $-1 < \text{Re}\,p < 0$, and the point $p = 0$ is a simple zero. In the second case, $\alpha = \alpha^*$. In this case also, there are no zeros in the interval $-1 < \text{Re}\,p < 0$, but the point $p = 0$ is a multiple zero. In the third case, $\alpha > \alpha^*$. Now a simple real zero appears in the interval $-1 < \text{Re}\,p < 0$, which we shall denote by $\lambda(\alpha)$. The zero $p = 0$ is again simple in this case. Besides, it should be noted that $-0.5 < \lambda(\alpha)$.

The above considerations can be used for isolating the required principal parts of Eq. (2.19). Since we shall be interested only in the principal part of the expression (2.19) for large values of r, we shall be considering the residues only at such points as are in the vicinity of the line $\text{Re}\,p = -1$. Consequently, in the first and second cases, we have to construct a residue only at the point $p = 0$, while for the third case, this should be done only at the point $\lambda(\alpha)$, which has been determined from the transcendental equation $G(p, \alpha) = 0$.

We normalize the loading region and the boundary condition by putting $\rho = r/a$, and $\varphi(\rho) = 2a^2 f(r)/M$. This gives

$$\int_0^1 \varphi(\rho)d\rho = 0, \qquad \int_0^1 \rho\varphi(\rho)d\rho = 1,$$

$$\bar{\varphi}(p) = \int_0^1 \rho^{p+1}\varphi(\rho)d\rho = \frac{2\bar{f}(p)}{Ma^p}.$$

It should be noted that all the results given below are valid only for $\rho > 1$.

In accordance with Eqs. (2.16) and (2.18), the expression for the component σ_r in the first case has the form

$$\sigma_r = \frac{2M \sin 2\theta}{a^2(\sin 2\alpha - 2\alpha \cos 2\alpha)\rho^2} + O(\rho^{-2}), \tag{2.22}$$

since the residue is taken only at the point $p = 0$.

The first term in the expression for, say, σ_r will be more cumbersome in the second case, since the zero at the point $p = 0$ is multiple (in order to construct a residue, we must multiply the integrand by p, differentiate with respect to p, and then put $p = 0$):

$$\sigma_r = \frac{M}{12a^2\alpha^2 \sin 2\alpha\rho^2}\{[12\bar{\varphi}'(0) - 12\ln\rho - 1]\sin 2\theta$$

$$+ 12\theta\cos 2\theta - 6\theta\cos 2\alpha\} + O(\rho^{-2}) \tag{2.23}$$

$$\left(\bar{\varphi}'(0) = \frac{d\bar{\varphi}}{dp}\bigg|_{p=0} = \int_0^1 \varphi(\rho)\rho\ln\rho\, d\rho\right).$$

The first term in the third case is determined from the residue at the point $\lambda(\alpha)$, while the second term is found from the residue at the point $p = 0$:

$$\sigma_r = \frac{M\bar{\varphi}(\lambda)[(\lambda + 4)\cos\lambda\alpha\sin(\lambda + 2)\theta - (\lambda + 2)\cos(\lambda + 2)\alpha\sin\lambda\theta]}{2a^2\{\sin 2\alpha - 2\alpha\cos[2(\lambda + 1)\alpha]\}\rho^{\lambda + 2}}$$

$$+ \frac{2M\sin 2\theta}{a^2(\sin 2\alpha - 2\alpha\cos 2\alpha)\rho^2} + O(\rho^{-2}). \tag{2.24}$$

Let us analyze the expressions (2.22)-(2.24) (the expressions for the other components have a similar structure). In the first place, it should be noted that these formulas can be used for the investigation of a set of boundary value problems only when the load and the region of its application remain unchanged, and the point under consideration tends to infinity. These formulas can also be used for obtaining a solution of the equivalent problem, where a point is fixed in a domain, and the region of application of force is decreased. In this case, the form of the boundary condition (dimensionless) is the same, i.e. the function $\varphi(\rho)$ remains unchanged. In the limit, this leads to the problem in which a moment is applied to the vertex of a wedge (see Sec. 7, Ch. 3, Vol. 1).

The expression for the component σ_r, which is directly obtained for this problem by applying the method of separation of variables given in Sec. 2, Ch. 4, is as follows:

$$\sigma_r = \frac{2M\sin 2\theta}{r^2(\sin 2\alpha - 2\alpha\cos 2\alpha)}. \tag{2.25}$$

A comparison of (2.22) and (2.25) shows that the limiting solution obtained by using rigorous methods is actually identical with the formal solution (2.25). Consequently, the distribution of stresses in the limit does not depend on the actual nature of the boundary condition, but is rather determined by the resultant moment. In the third case, the expression (2.24) contains terms which appear in the solution (2.25). However, these are not significant, and the state of stress in the limit will be determined by the first term only. It is important to note that this term depends on the function φ, and hence, on the nature of the actual applied load. Thus, we arrive at an example which is in contradiction with the generally accepted formulation of Saint-Venant's principle.

Let us consider another case, where

$$\bar{\varphi}(\lambda) = \int_0^1 \rho^{\lambda+1}\varphi(\rho)d\rho = 0$$

and the first term in (2.24) vanishes. The solution (2.25) is found to be true in this case also. In the second case, however, we cannot speak of a limit transition in view of the presence of a term containing $\ln \rho$.

The above discussion helps in explaining the well-known Carothers paradox according to which $\sigma_r \to \infty$ as $\alpha \to \alpha^*$ in the solution (2.25), since in this case (2.25) is not the principal part of the solution.

In conclusion of this section, it should be mentioned that a solution of the problem for a wedge with other boundary conditions ($f_1 \neq -f_2$) leads to a decrease in the value of the angle α for which the formula (2.25) is applicable.

Section 3
Axially Symmetric Problem for a Plate with a Circular Cut

Suppose that in a plate of thickness $2h$, we have a cut of radius a in the middle plane $z = 0$. We assume that only normal pressures, which are the same on different sides, are applied to this cut. The boundaries of the plate are assumed to be free of stresses. Thus, the plane $z = 0$ is the symmetry plane.

The boundary conditions have the following form:

$$\sigma_z(\rho, \pm h) = 0, \qquad \tau_{\rho z}(\rho, \pm h) = 0, \tag{3.1}$$

$$\sigma_z(\rho, 0) = p(\rho), \qquad \tau_{\rho z}(\rho, 0) = 0 \quad (\rho < a). \tag{3.2}$$

Besides, it follows from symmetry considerations that

$$\tau_{\rho z}(\rho, 0) = 0 \qquad (\rho < a), \tag{3.3}$$

$$u_z(\rho, 0) = 0 \qquad (\rho > a). \tag{3.4}$$

In order to solve this problem, we start with the Papkovich-Neuber representations which, on account of axial symmetry, contain two harmonic functions φ and ψ_z. We shall first give the expressions for the components of displacements and

stresses, which we shall be requiring later (see Eqs. (5.45) and (5.46), Ch. 3, Vol. 1):

$$u_z = (3 - 4\nu)\psi - z\frac{\partial\psi}{\partial z} - \frac{\partial\varphi}{\partial z},$$

$$(2G)^{-1}\sigma_z = 2(1-\nu)\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial z^2} - \frac{\partial^2\varphi}{\partial z^2},$$

$$(2G)^{-1}\tau_{\rho z} = \frac{\partial}{\partial\rho}\left[(1-2\nu)\psi - z\frac{\partial\psi}{\partial z} - \frac{\partial\varphi}{\partial z}\right].$$

Here, the index of the function φ_z has been omitted.

We start with the representations of the harmonic functions φ and ψ in the form of the integrals [45]

$$\frac{\partial\varphi}{\partial z} = \int_0^\infty [A_1\cosh\lambda(h-z) + B_1\sinh\lambda(h-z)]J_0(\lambda\rho)\frac{d\lambda}{\sinh\lambda h}, \tag{3.5}$$

$$\psi = \int_0^\infty [A_2\cosh\lambda(h-z) + B_2\sinh\lambda(h-z)]J_0(\lambda\rho)\frac{d\lambda}{\sinh\lambda h}, \tag{3.6}$$

where $A_1(\lambda)$, $B_1(\lambda)$, $A_2(\lambda)$, and $B_2(\lambda)$ are functions to be determined later.

It follows from the conditions (3.1) that the following equalities are valid:

$$A_1(\lambda) = (1-2\nu)A_2(\lambda) + \lambda h B_2(\lambda), \tag{3.7}$$

$$B_1(\lambda) = \lambda h A_2(\lambda) + 2(1-\nu)B_2(\lambda). \tag{3.8}$$

The condition $\tau_{\rho z}(\rho, 0) = 0$ leads to another dependence between these two functions, which can be written in a compact form with the help of Eqs. (3.7) and (3.8):

$$\lambda h A_2(\lambda) + (1 + \lambda h\coth\lambda h)B_2(\lambda) = 0. \tag{3.9}$$

The relations (3.7)-(3.9) can be used to represent any three of these functions in terms of one. We express all of them in terms of $B_2(\lambda)$ and turn to the conditions (3.2) and (3.4) taking, of course, the representations (3.5) and (3.6) into account:

$$\int_0^\infty B_2(\lambda)\frac{\lambda h + \cosh\lambda h\sinh\lambda h}{\lambda h\sinh^2\lambda h}J_0(\lambda\rho)d\lambda = 0 \quad (\rho > a), \tag{3.10}$$

$$\int_0^\infty \lambda B_2(\lambda)\frac{\sinh^2\lambda h - \lambda^2h^2}{\lambda h\sinh^2\lambda h}J_0(\lambda\rho)d\lambda = p(\rho)(2G)^{-1} = f(\rho) \quad (\rho < a). \tag{3.11}$$

These equations can be written in a more compact form:

$$\int_0^\infty B(\lambda)J_0(\lambda\rho)d\lambda = 0 \qquad (\rho > a), \tag{3.10'}$$

$$\int_0^\infty \lambda[1 - g(\lambda)]B(\lambda)J_0(\lambda\rho)d\lambda = f(\rho) \qquad (\rho < a), \tag{3.11'}$$

where

$$B(\lambda) = B_2(\lambda)\,\frac{\lambda h + \cosh\lambda h \sinh\lambda h}{\lambda h \sinh^2\lambda h},$$

$$g(\lambda) = \frac{\lambda h(1 + \lambda h) + e^{-\lambda h}\sinh\lambda h}{\cosh\lambda h \sinh\lambda h + \lambda h}.$$

It should be noted that if we are considering the entire space with a cut, we should start with the representations

$$\frac{\partial\varphi}{\partial z} = \int_0^\infty A(\lambda)e^{-\lambda z}J_0(\lambda\rho)d\lambda, \tag{3.5'}$$

$$\psi = \int_0^\infty B(\lambda)e^{-\lambda z}J_0(\lambda\rho)d\lambda \tag{3.6'}$$

(for the sake of definiteness, all the constructions take place in the upper half-space). This leads to the same pair of equations (3.10) and (3.11), but for $g(\lambda) = 0$.

If we introduce the representation

$$B(\lambda) = \int_0^a C(t)\sin\lambda t\,dt, \tag{3.12}$$

we find that Eq. (3.10′) is identically satisfied. As a matter of fact,

$$\int_0^\infty B(\lambda)J_0(\lambda\rho)d\lambda = \int_0^a C(t)dt \int_0^\infty J_0(\lambda\rho)\sin\lambda t\,d\lambda. \tag{3.13}$$

The last integral is called the Weber integral [46]. The following equality is valid for this integral:

$$\int_0^\infty J_0(\lambda\rho)\sin\lambda t\,d\lambda = \begin{cases} 0, & 0 \leqslant t < \rho, \\ (t^2 - \rho^2)^{-1/2}, & 0 < \rho < t. \end{cases} \tag{3.14}$$

Since in our case $t < a$, and $\rho > a$, we get $\rho \geqslant t$, which proves the validity of the above statement.

We substitute (3.12) into (3.11), and integrate with respect to ρ. For this purpose, we first use the identity

$$\lambda\rho J_0(\lambda\rho) = \frac{d}{d\rho}\,[\rho J_1(\lambda\rho)]. \tag{3.15}$$

Consequently, we get

$$\rho \int_0^a C(t)dt \int_0^\infty J_1(\lambda\rho)\sin\lambda t\, d\lambda - \rho \int_0^a C(t)dt \int_0^\infty g(\lambda)J_1(\lambda\rho)\sin\lambda t\, d\lambda$$

$$= \int_0^\rho \rho f(\rho)d\rho = f_1(\rho). \quad (3.16)$$

Using the equality

$$\int_0^\infty J_1(\lambda\rho)\sin\lambda t\, d\lambda = \begin{cases} 0, & \rho < t, \\ \dfrac{t}{\rho}(\rho^2 - t^2)^{-1/2}, & \rho > t, \end{cases} \quad (3.17)$$

we can transform (3.16) as follows:

$$\int_0^\rho \frac{tC(t)dt}{\sqrt{\rho^2 - t^2}} - \int_0^a C(t)dt \int_0^\infty g(\lambda)\sin\lambda t J_1(\lambda\rho)d\lambda = f_1(\rho). \quad (3.18)$$

Next, we use the integral representation for the function

$$J_1(\lambda\rho) = \frac{2}{\pi}\int_0^{\pi/2} \sin\theta \sin(\lambda\rho\sin\theta)d\theta \quad (3.19)$$

and make the substitution

$$t = \rho\sin\theta. \quad (3.20)$$

As a result, we get the equation

$$\int_0^{\pi/2} \left[C(\rho\sin\theta) - \frac{2}{\pi}\int_0^a C(t)dt \int_0^\infty g(\lambda)\sin\lambda t \sin(\lambda\rho\sin\theta)d\lambda \right] \times \rho\sin\theta\, d\theta = f_1(\rho), \quad (3.21)$$

which may be symbolically written in the form

$$\int_0^{\pi/2} \Phi(\rho\sin\theta)\rho\sin\theta\, d\theta = \int_0^{\pi/2} \Phi_1(\rho\sin\theta)d\theta = f_1(\rho). \quad (3.21')$$

This is the familiar Schloemilch equation (2.41), Ch. 1, Vol. 1. Its continuous solution is given by the formula

$$\Phi_1(\rho) = \frac{2}{\pi}\left[f_1(0) + \rho \int_0^{\pi/2} f_1'(\rho\sin\theta)d\theta \right]. \quad (3.22)$$

Considering that in our case $f_1(0) = 0$, we get the equality

$$C(t) - \frac{2}{\pi}\int_0^a \left[\int_0^\infty g(\lambda)\sin\lambda t \sin\lambda y\, d\lambda\right] C(y)dy = \frac{2}{\pi}\int_0^{\pi/2} f_1'(t\sin\theta)d\theta, \tag{3.23}$$

which is a Fredholm integral equation of the second kind. Solving (generally speaking, numerically) this equation and carrying out the inverse transformation, we can find all the parameters in which we are interested.

For an infinite space with a cut, $g(\lambda) = 0$. Hence the equation degenerates and its solution is identical with the right-hand side:

$$C(t) = \frac{2}{\pi}\int_0^{\pi/2} f_1'(t\sin\theta)d\theta. \tag{3.24}$$

We shall now write down the expressions for displacements $u_z(\rho, 0)$ for $\rho < a$, for the stresses σ_z on the extension of the cut for $\rho > a$ in the simplest case $p = p_0$:

$$u_z = \frac{4p_0(1-\nu^2)}{\pi E}(a^2 - \rho^2)^{1/2}, \tag{3.25}$$

$$\sigma_z = -\frac{2p_0}{\pi}\left[\arcsin\frac{a}{\rho} - a(\rho^2 - a^2)^{-1/2}\right]. \tag{3.26}$$

This leads to an expression for the stress intensity factor (see Sec. 9, Ch. 3, Vol. 1):

$$K_I = \frac{\sqrt{2}p_0 a^{1/2}}{\pi}. \tag{3.27}$$

Section 4
The Plane Problem on the Concentrated Action of an Impulsive Force (Lamb's Problem)

Suppose that the boundary of a half-space $y \geqslant 0$ which is at rest for $t < 0$ is subjected at the moment $t = 0$ to a concentrated impulsive force which is distributed uniformly along the line $y = x = 0$:

$$\sigma_y = -k\delta(x)\delta(t) \qquad (k = \text{const}, \quad y = 0). \tag{4.1}$$

Since the initial conditions as well as the boundary conditions are independent of z, the problem will be a plane one. Consequently, the system of equations of motion of the medium and the initial and boundary conditions can be written as follows:

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial\tau^2}, \qquad \Delta\Psi = \gamma^2\frac{\partial^2\Psi}{\partial\tau^2} \quad \left(\tau = at, \quad \gamma = \frac{a}{b}\right), \tag{4.2}$$

$$\sigma_y = (a^2 - 2b^2)\rho\Delta\Phi + 2b^2\rho\left(\frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x\partial y}\right) = -\delta(x)\delta(\tau)k_0 \quad (y = 0), \tag{4.3}$$

$$\tau_{xy} = b^2\rho\left(2\frac{\partial^2\Phi}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x^2}\right) = 0 \qquad (y = 0),$$

$$\Phi = \Psi = \frac{\partial\Phi}{\partial\tau} = \frac{\partial\Psi}{\partial\tau} = 0 \qquad (\tau = 0) \tag{4.4}$$

(while writing the boundary condition (4.3), we have made use of the property $\delta(t) = a\delta(at) = a\delta(\tau)$, and $k_0 = k/a$).

Applying the Laplace transformation in τ, and then the Fourier transformation in x to the system (4.2)-(4.4), we get

$$\frac{d^2\bar\Phi^*}{dy^2} = (p^2 + q^2)\bar\Phi^*,$$
$$\frac{d^2\bar\Psi^*}{dy^2} = (\gamma^2p^2 + q^2)\bar\Psi^*; \tag{4.5}$$

$$(\gamma^2 - 2)p^2\bar\Phi^* + 2\left(\frac{d^2\bar\Phi^*}{dy^2} + iq\frac{d\bar\Psi^*}{dy}\right) = -\frac{k_0}{\mu}$$

$$(y = 0,\quad \mu = b^2\rho), \tag{4.6}$$

$$\frac{d^2\bar\Psi^*}{dy^2} + q^2\bar\Psi^* - 2iq\frac{d\bar\Phi^*}{dy} = 0 \qquad (y = 0).$$

Here,

$$\bar f(p,x,y) = \int_0^\infty f(\tau,x,y)e^{-p\tau}d\tau, \qquad \operatorname{Re} p > 0,$$

$$\bar f^*(p,q,y) = \int_{-\infty}^{\infty} \bar f(p,x,y)e^{ixq}dx, \quad \operatorname{Im} q = 0 \tag{4.7}$$

$$(f = \Phi, \Psi).$$

Solving Eqs. (4.6) for $\bar\Phi^*$ and $\bar\Psi^*$, we get

$$\bar\Phi^* = Ae^{-\sqrt{p^2+q^2}y} + Be^{\sqrt{p^2+q^2}y},$$
$$\bar\Psi^* = Ce^{-\sqrt{\gamma^2p^2+q^2}y} + De^{\sqrt{\gamma^2p^2+q^2}y} \tag{4.8}$$

In order to isolate the single-valued branches of the radicals $\sqrt{p^2 + q^2}$ and $\sqrt{\gamma^2p^2 + q^2}$ in the plane q, we make cuts from the points $\pm pi$ and $\pm\gamma pi$ to infinity along the rays $\arg q = \arg p \pm \pi/2$ (Fig. 49) and assume that, for $q = 0$, $\sqrt{p^2 + q^2} = p$, $\sqrt{\gamma^2p^2 + q^2} = \gamma p$. We can then easily verify that $\operatorname{Re}\sqrt{p^2 + q^2} > 0$ and $\operatorname{Re}\sqrt{\gamma^2p^2 + q^2} > 0$ for $\operatorname{Im} q = 0$ and $\operatorname{Re} p > 0$ (it can be easily done by observing the change in the arguments φ_0, φ_0' and φ_1, φ_1' as the point q moves along the real axis in Fig. 49).

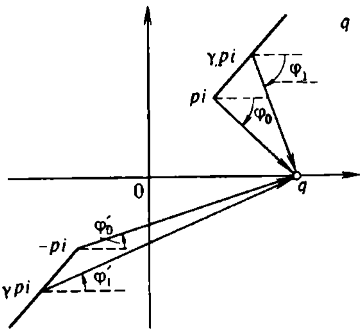

Fig. 49. A system of cuts.

Since the medium was at rest at $\tau < 0$, and the perturbations appear in a bounded domain, the transforms $\bar{\Phi}^*$ and $\bar{\Psi}^*$ must decrease as $y \to \infty$. Consequently, we must put $B = D = 0$ in (4.8). After this, we can find A and C by substituting $\bar{\Phi}^*$ and $\bar{\Psi}^*$ from (4.8) into (4.6):

$$A = -\frac{\gamma^2 p^2 + 2q^2}{R(p,q)}\frac{k_0}{\mu}, \qquad C = \frac{2iq\sqrt{p^2+q^2}}{R(p,q)}\frac{k_0}{\mu}. \tag{4.9}$$

Here,

$$R(p,q) = (\gamma^2p^2 + 2q^2)^2 - 4q^2\sqrt{(\gamma^2p^2+q^2)(p^2+q^2)}.$$

Since the transforms of the components of the displacement vector $\mathbf{u}(u, v)$ are connected with the transforms of the potentials Φ and Ψ through the relations

$$\bar{u}^* = -iq\bar{\Phi}^* + \frac{d\bar{\Psi}^*}{dy}, \qquad \bar{v}^* = \frac{d\bar{\Phi}^*}{dy} + iq\bar{\Psi}^*, \tag{4.10}$$

we get the following expressions for the transforms $\bar{u}$ and $\bar{v}$ with the help of Eqs. (4.8)-(4.10):

$$\bar{u}(p,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iqx}\bar{u}^*\, dq.$$

$$\bar{v}(p,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iqx}\bar{v}^*\, dq; \tag{4.11}$$

$$\bar{u}^*(p,q,y) = \frac{iqk_0}{\mu R(p,q)}\Big[(\gamma^2p^2 + 2q^2)e^{-\sqrt{p^2+q^2}\,y} - 2\sqrt{(p^2+q^2)(\gamma^2p^2+q^2)}\,e^{-\sqrt{\gamma^2p^2+q^2}\,y}\Big],$$

$$\bar{v}^*(p,q,y) = \frac{k_0\sqrt{p^2+q^2}}{\mu R(p,q)}\Big[(\gamma^2p^2+2q^2)e^{-\sqrt{p^2+q^2}\,y} - 2q^2e^{-\sqrt{\gamma^2p^2+q^2}\,y}\Big].$$

The formulas (4.11) give the values of $\bar{u}$ and $\bar{v}$ which are the Laplace transforms of the required solution of the problem for the components of the displacement vector. In order to obtain the expressions for the originals, we must apply the inverse Laplace transform in p to the expressions (4.11). For this purpose, we use Cagniard's method (as modified by de Hoop [47]), according to which the integrals of the inverse Fourier transform in q, given by the expressions for $\bar{u}$ and $\bar{v}$ in (4.11), are transformed into integrals of the type of the Laplace transform in τ, i.e. into integrals of the form

$$\int_0^\infty f(\tau, x, y) e^{-p\tau} d\tau. \tag{4.12}$$

This leads at once to the original of this expression which must have the form of the integrand $f(\tau, x, y)$. In order to reduce the expressions for $\bar{u}$ and $\bar{v}$ in (4.11) to the form (4.12), we can represent them as follows:

$$\bar{u} = \bar{u}_1 + \bar{u}_2, \qquad \bar{v} = \bar{v}_1 + \bar{v}_2; \tag{4.13}$$

$$\bar{u}_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k_0 iq(\gamma^2 p^2 + 2q^2)}{\mu R(p, q)} e^{-\sqrt{p^2 + q^2}y - iqx} dq,$$

$$\bar{u}_2 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k_0 2iq\sqrt{(p^2 + q^2)(\gamma^2 p^2 + q^2)}}{\mu R(p, q)} e^{-\sqrt{\gamma^2 p^2 + q^2}y - iqx} dq,$$

$$\bar{v}_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{k_0 \sqrt{p^2 + q^2}}{\mu R(p, q)} (\gamma^2 p^2 + 2q^2) e^{-\sqrt{p^2 + q^2}y - iqx} dq,$$

$$\bar{v}_2 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2k_0 q^2 \sqrt{p^2 + q^2}}{\mu R(p, q)} e^{-\sqrt{\gamma^2 p^2 + q^2}y - iqx} dq.$$

Changing the integration variable q in (4.13) as per formula $q = -ips$, we get

$$\begin{aligned}
\bar{u}_1 &= -\frac{ik_0}{2\pi\mu} \int_L \frac{s(\gamma^2 - 2s^2)}{R(s)} e^{-p(y\sqrt{1 - s^2} + sx)} ds, \\
\bar{u}_2 &= \frac{ik_0}{\pi\mu} \int_L \frac{s\sqrt{(1 - s^2)(\gamma^2 - s^2)}}{R(s)} e^{-p(y\sqrt{\gamma^2 - s^2} + sx)} ds, \\
\bar{v}_1 &= -\frac{ik_0}{2\pi\mu} \int_L \frac{\sqrt{1 - s^2}(\gamma^2 - 2s^2)}{R(s)} e^{-p(y\sqrt{1 - s^2} + sx)} ds, \\
\bar{v}_2 &= \frac{ik_0}{\pi\mu} \int_L \frac{s^2\sqrt{1 - s^2}}{R(s)} e^{-p(y\sqrt{\gamma^2 - s^2} + sx)} ds.
\end{aligned} \tag{4.14}$$

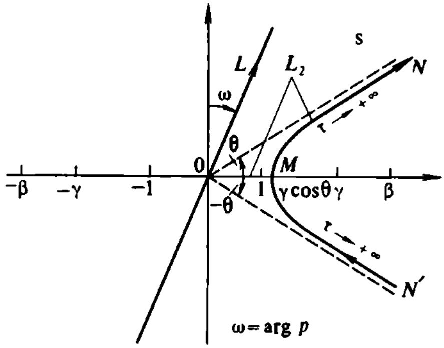

Fig. 50. The L and L_2 contours.

The contour L is shown in Fig. 50. Putting $\sqrt{p^2+q^2} = p\sqrt{1-s^2}$ and $\sqrt{\gamma^2 p^2 + q^2} = p\sqrt{\gamma^2 - s^2}$, we can also determine the single-valued branches of the radicals $\sqrt{1-s^2}$ and $\sqrt{\gamma^2-s^2}$ (taking into consideration the branches $\sqrt{p^2+q^2}$ and $p\sqrt{\gamma^2-s^2}$ chosen earlier) as follows: $\sqrt{1-s^2} = 1$ and $\sqrt{\gamma^2-s^2} = \gamma$ for $s = 0$, and the cuts for isolating these single-valued branches now pass along the real semi-axes from the points $s = \pm 1$ and $s = \pm\gamma$ to infinity. The complex function

$$R(s) = (\gamma^2 - 2s^2)^2 + 4s^2\sqrt{(\gamma^2 - s^2)(1 - s^2)}$$

is the left-hand side of the Rayleigh equation (7.39), Ch. 5:

$$R(s) = 0.$$

It is well known [34] that in the complex plane s, this equation has two real roots: $s_{1,2} = \pm\beta$, where $\beta > \gamma$ (in particular, for $\lambda = \mu$, we get $\beta = \gamma(2 - 2/\sqrt{3})^{-1/2} \approx \gamma/0.9194$). We now deform the integration contour L in the formulas (4.14) for $\bar{u}_1$ and $\bar{v}_1$ into such a contour L_1 that the expression $y\sqrt{1-s^2} + sx$ is real along this contour. Similarly, the contour L should be deformed for $\bar{u}_2$ and $\bar{v}_2$ in (4.14) into such a contour L_2 that the expression $y\sqrt{\gamma^2-s^2} + sx$ is real along this contour.

We shall first determine the contour L_2. For this purpose, we put

$$y\sqrt{\gamma^2 - s^2} + sx = \tau, \tag{4.15}$$

where τ is real. In this equation, τ should be considered as a parameter along the path L_2 of integration in the complex plane s. Later, we shall replace the integration variable s by τ and consider τ as time. For this reason we have chosen the above notation for the quantity $y\sqrt{\gamma^2-s^2} + sx$.

Solving Eq. (4.15) for s, we get

$$s = \frac{\tau x \pm y\sqrt{\gamma^2(x^2+y^2) - \tau^2}}{x^2 + y^2} \qquad (|\tau| < \gamma\sqrt{x^2+y^2}), \tag{4.16'}$$

$$s = \frac{\tau x \pm iy\sqrt{\tau^2 - \gamma^2(x^2 + y^2)}}{x^2 + y^2} \qquad (|\tau| > \gamma\sqrt{x^2 + y^2}). \tag{4.16''}$$

Here, the arithmetic values of the radicals have been taken. With changing τ, the expressions (4.16′) and (4.16″) describe certain contours $s = s(\tau)$ in the plane s. In accordance with the branch of the radical $\sqrt{\gamma^2 - s^2}$ chosen in (4.15) (i.e. in view of the fact that $\tau = \gamma y$ for $s = 0$), we must choose the minus sign in (4.16′) and retain both signs in (4.16″) in order to obtain a contour described by the expression (4.15) for $\tau > 0$ and $x \geqslant 0$ (it is sufficient to consider the case $x \geqslant 0$, taking into account the fact that v is even and u is odd in x). In Fig. 50, the solid curve in the region $\operatorname{Re} s > 0$ shows the contour L_2 consisting of the segment OM and curve $N'MN$ which is given by the expression ($x \geqslant 0$)

$$\begin{aligned} s &= \frac{\tau x - y\sqrt{\gamma^2(x^2 + y^2) - \tau^2}}{x^2 + y^2} \qquad (\gamma y < \tau < \gamma\sqrt{x^2 + y^2}), \\ s &= \frac{\tau x \pm iy\sqrt{\tau^2 - \gamma^2(x^2 + y^2)}}{x^2 + y^2} \qquad (\tau > \gamma\sqrt{x^2 + y^2}). \end{aligned} \tag{4.17}$$

The plus and minus signs in the second expression in this equation correspond to the curves in the upper and lower half-planes s respectively. Putting $x = r\cos\theta$ and $y = r\sin\theta$ $(0 < \theta < \pi/2)$ in the formulas (4.17), we get

$$s(\tau, r, \theta) = \begin{cases} \dfrac{1}{r}\,(\tau\cos\theta - \sqrt{\gamma^2 r^2 - \tau^2}\sin\theta) & \text{for } \tau < \gamma r, \\ \dfrac{1}{r}\,(\tau\cos\theta \pm i\sqrt{\tau^2 - \gamma^2 r^2}\sin\theta) & \text{for } \tau > \gamma r. \end{cases}$$

We shall assume that as $\tau \to +\infty$, r and θ remain fixed. Then, we find that $s \to \tau e^{\pm i\theta}/r$ and, hence, as shown in Fig. 50, the curve L_2 is bounded by the asymptotes emerging from the origin $s = 0$ at an angle $\pm\theta$ to the real axis (for $\tau \to +\infty$).

As $s \to \infty$, the expression $p(y\sqrt{\gamma^2 - s^2} + sx)$ can be represented in the form

$$p(y\sqrt{\gamma^2 - s^2} + sx) = \begin{cases} |p|\,|s|\,re^{i(\omega + \theta + \varphi)} & (\operatorname{Re} s < 0), \\ |p|\,|s|\,re^{i(\omega - \theta + \varphi)} & (\operatorname{Re} s > 0), \end{cases}$$

where

$$p = |p|e^{i\omega} \quad \left(|\omega| < \frac{\pi}{2}\right), \qquad s = |s|e^{i\varphi}.$$

Since

$$\min\left(-\theta, -\frac{\pi}{2} - \omega\right) \leqslant \varphi \leqslant \max\left(-\theta, -\frac{\pi}{2} - \omega\right) \quad (\operatorname{Re} s < 0),$$

$$\min\left(\theta, \frac{\pi}{2} - \omega\right) \leqslant \varphi \leqslant \max\left(\theta, \frac{\pi}{2} - \omega\right) \qquad (\operatorname{Re} s > 0),$$

we get

$$-\frac{\pi}{2} < \min\left(-\frac{\pi}{2}+\theta, \omega\right) \leqslant (\varphi+\theta+\omega)$$

$$\leqslant \max\left(-\frac{\pi}{2}+\theta, \omega\right) < \frac{\pi}{2} \quad (\operatorname{Re} s < 0),$$

$$(4.18)$$

$$-\frac{\pi}{2} < \min\left(\frac{\pi}{2}-\theta, \omega\right) \leqslant (\varphi+\omega-\theta)$$

$$\leqslant \max\left(\frac{\pi}{2}-\theta, \omega\right) < \frac{\pi}{2} \quad (\operatorname{Re} s > 0).$$

Consequently, the exponent $-p(\sqrt{\gamma^2 - s^2}y + sx)$ has a negative real part for $|s| \to \infty$ in the region between the curves L and L_2. Hence, with the help of Jordan's lemma (see Sec. 4, Ch. 1, Vol. 1), we can deform the integration contour L in the expressions (4.14) for $\bar{u}_2$ and $\bar{v}_2$ into the contour L_2 which is traversed along $N'MOMN$, the segment OM of the real axis is passed twice in the opposite directions, along the upper and lower edges respectively. Then, deforming the contour L into L_2 and making a change of variable $\tau = y\sqrt{\gamma^2 - s^2} + sx$, we can successively convert $\bar{u}_2$ and $\bar{v}_2$ in Eqs. (4.14) into the following form:

$$\bar{u}_2 = i\int_{L_2} f_2 ds$$

$$= i\int_{\bar{l}_2} f_2 ds + i\int_{l_2} f_2 ds = i\int_{+\infty}^{\gamma y} f_2 \frac{ds}{d\tau} d\tau + i\int_{\gamma y}^{+\infty} f_2 \frac{ds}{d\tau} d\tau$$

$$= -\frac{2k_0}{\mu}\int_{\gamma y}^{+\infty} \operatorname{Im}\left[\frac{s\sqrt{(1-s^2)(\gamma^2-s^2)}}{R(s)}\frac{ds}{d\tau}\right] e^{-p\tau} d\tau, \tag{4.19}$$

$$\bar{v}_2 = i\int_{L_2} F_2 ds$$

$$= i\int_{\bar{l}_2} F_2 ds + i\int_{l_2} F_2 ds = i\int_{+\infty}^{\gamma y} F_2 \frac{ds}{d\tau} d\tau + i\int_{\gamma y}^{+\infty} F_2 \frac{ds}{d\tau} d\tau$$

$$= \frac{2k_0}{\pi\mu}\int_{\gamma y}^{+\infty} \operatorname{Im}\left[\frac{s^2\sqrt{1-s^2}}{R(s)}\frac{ds}{d\tau}\right] e^{-p\tau} d\tau, \tag{4.20}$$

$$f_2 = \frac{k_0}{\pi\mu} \frac{s\sqrt{(1-s^2)(\gamma^2-s^2)}}{R(s)} e^{-p(y\sqrt{\gamma^2-s^2}+sx)},$$

$$F_2 = -\frac{k_0}{\pi\mu} \frac{s^2\sqrt{1-s^2}}{R(s)} e^{-p(y\sqrt{\gamma^2-s^2}+sx)}.$$

Whille obtaining the expressions (4.19) and (4.20), we have assumed that the branches l_2 and $\bar{l}_2$ of the curve L_2 (l_2 and $\bar{l}_2$ represent the curves OMN and $N'MO$ respectively) are symmetric with respect to the real axis. Besides, we have considered the fact that the integrands

$$\frac{s\sqrt{(1-s^2)(\gamma^2-s^2)}}{R(s)} \frac{ds}{d\tau}, \quad \frac{s^2\sqrt{1-s^2}}{R(s)} \frac{ds}{d\tau}$$

are complex conjugate at the complex-conjugate points s and $\bar{s}$ on l_2 and $\bar{l}_2$ respectively. Moreover, the function $y\sqrt{\gamma^2 - s^2} + sx$ assumes equal real values of τ at these points.

It should be also noted that in the final expressions for $\bar{u}_2$ and $\bar{v}_2$ in (4.19) and (4.20), the integrands

$$\operatorname{Im}\left[\frac{s^2\sqrt{1-s^2}}{R(s)} \frac{ds}{d\tau}\right], \quad \operatorname{Im}\left[\frac{s\sqrt{(1-s^2)(\gamma^2-s^2)}}{R(s)} \frac{ds}{d\tau}\right]$$

are determined from the condition that s lies on the contour l_2 (Figs. 51 and 52), whose equation is of the form

$$\begin{aligned} s &= \frac{\tau x - y\sqrt{\gamma^2(x^2+y^2)-\tau^2}}{x^2+y^2} \quad (\gamma y < \tau < \gamma\sqrt{x^2+y^2}), \\ s &= \frac{\tau x + iy\sqrt{\tau^2-\gamma^2(x^2+y^2)}}{x^2+y^2} \quad (\tau > \gamma\sqrt{x^2+y^2}). \end{aligned} \tag{4.21}$$

The integrals for $\bar{u}_1$ and $\bar{v}_1$ in (4.14) can be transformed in an exactly similar way. Here, the contours L_1 and l_1 can be obtained from L_2 and l_2 respectively by putting $\gamma = 1$ in the formulas (4.15) and (4.21). As a result, we get

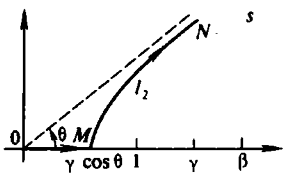

Fig. 51. The l_2 contour ($\gamma\cos\theta < 1$).

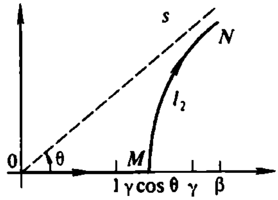

Fig. 52. The l_2 contour ($\gamma \cos\theta > 1$).

$$\bar{u}_1 = \frac{k_0}{\pi\mu}\int_y^\infty \operatorname{Im}\left[\frac{s(\gamma^2 - 2s^2)}{R(s)}\frac{ds}{d\tau}\right]e^{-p\tau}d\tau, \tag{4.22}$$

$$\bar{v}_1 = \frac{k_0}{\pi\mu}\int_y^\infty \operatorname{Im}\left[\frac{(\gamma^2 - 2s^2)\sqrt{1 - s^2}}{R(s)}\frac{ds}{d\tau}\right]e^{-p\tau}d\tau, \tag{4.23}$$

where s lies on the contour l_1 (Fig. 53) whose equation is of the form

$$\begin{aligned} s &= \frac{\tau x - y\sqrt{(x^2 + y^2) - \tau^2}}{x^2 + y^2} \qquad (y < \tau < \sqrt{x^2 + y^2}), \\ s &= \frac{\tau x + iy\sqrt{\tau^2 - (x^2 + y^2)}}{x^2 + y^2} \qquad (\tau > \sqrt{x^2 + y^2}). \end{aligned} \tag{4.24}$$

It is clear from the formulas (4.22) and (4.23) that the integrands in them actually vanish for real values of s satisfying the condition $s < 1$, i.e. in fact for $s < \cos\theta$ or, in the case of variable τ, for $\tau < r$. Consequently, the expressions (4.22) and (4.23) may be written in the following form:

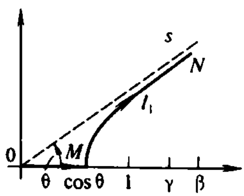

Fig. 53. The l_1 contour.

$$\bar{u}_1 = \frac{k_0}{\pi\mu} \int_0^\infty \mathrm{Im} \left[\frac{s(\gamma^2 - 2s^2)}{R(s)} \frac{ds}{d\tau} \right] H(\tau - r) e^{-p\tau} d\tau,$$

$$\bar{v}_1 = \frac{k_0}{\pi\mu} \int_0^\infty \mathrm{Im} \left[\frac{(\gamma^2 - 2s^2)\sqrt{1 - s^2}}{R(s)} \frac{ds}{d\tau} \right] H(\tau - r) e^{-p\tau} d\tau \tag{4.25}$$

$$\left(H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \right),$$

where the function s is defined by the formulas (4.24).

Similarly, by considering the cases $\gamma\cos\theta < 1$ (see Fig. 51) and $\gamma\cos\theta > 1$ (see Fig. 52), we get the following expressions for $\bar{u}_2$ and $\bar{v}_2$:

$$\bar{u}_2 = -\frac{2k_0}{\pi\mu} \int_0^\infty \mathrm{Im} \left[\frac{s\sqrt{(1 - s^2)(\gamma^2 - s^2)}}{R(s)} \frac{ds}{d\tau} \right] \times \begin{Bmatrix} H(\tau - r\gamma) \\ H(\tau - x - y\sqrt{\gamma^2 - 1}) \end{Bmatrix} e^{-p\tau} d\tau,$$

$$\bar{v}_2 = \frac{2k_0}{\pi\mu} \int_0^\infty \mathrm{Im} \left[\frac{s^2\sqrt{1 - s^2}}{R(s)} \frac{ds}{d\tau} \right] \times \begin{Bmatrix} H(\tau - r\gamma) \\ H(\tau - x - y\sqrt{\gamma^2 - 1}) \end{Bmatrix} e^{-p\tau} d\tau, \tag{4.26}$$

where the upper and lower rows in each of the above expressions strictly belong to the cases $\gamma\cos\theta < 1$ and $\gamma\cos\theta > 1$ respectively, and the function s is defined by the expressions (4.21).

Considering the formulas (4.25) and (4.26) once again, we find that the representations for u_1, v_1, u_2, and v_2 are now given in the form of integrals which are the Laplace transforms in τ of the real functions appearing in the integrands. Hence, these integrand functions are the required originals.

Thus, applying the Laplace inverse transformation in p to (4.25) and (4.26), we finally get

$$u = u_1 + u_2,$$

$$u_1 = \frac{k_0}{\pi\mu} \mathrm{Im} \left[\frac{s(\gamma^2 - 2s^2)}{R(s)} \frac{ds}{d\tau} \right] H(\tau - r), \tag{4.27}$$

$$u_2 = -\frac{2k_0}{\pi\mu} \mathrm{Im} \left[\frac{s\sqrt{(1 - s^2)(\gamma^2 - s^2)}}{R(s)} \frac{ds}{d\tau} \right] \begin{Bmatrix} H(\tau - r\gamma) \\ H(\tau - x - y\sqrt{\gamma^2 - 1}) \end{Bmatrix};$$

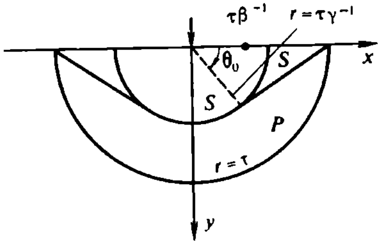

Fig. 54. The wavefronts.

$$v = v_1 + v_2,$$

$$v_1 = \frac{k_0}{\pi\mu}\,\mathrm{Im}\left[\frac{(\gamma^2 - 2s^2)\sqrt{1-s^2}}{R(s)}\,\frac{ds}{d\tau}\right] H(\tau - r), \tag{4.28}$$

$$v_2 = \frac{2k_0}{\pi\mu}\,\mathrm{Im}\left[\frac{s^2\sqrt{1-s^2}}{R(s)}\,\frac{ds}{d\tau}\right]\begin{Bmatrix} H(\tau - r\gamma) \\ H(\tau - x - y\sqrt{\gamma^2 - 1}) \end{Bmatrix},$$

where s is given by (4.24) for the formulas for u_1 and v_1, and by (4.21) for the formulas for u_2 and v_2.

Let us consider the geometrical pattern of the wavefronts in the region $y \geqslant 0$ (Fig. 54). The expressions u_1 and v_1 represent the contribution from the longitudinal wave P, whose wavefront is given by the equation $r = \tau$, while the expressions u_2 and v_2 are given by the transverse wave S, containing a wave with the circular wavefront $r = \tau\gamma^{-1}$, and a leading transverse wave with a straight wavefront $\tau - x - y\sqrt{\gamma^2 - 1} = 0$. The leading transverse wave S is generated by the interaction of the travelling longitudinal wave P with the free surface. The wavefront of the leading transverse wave is tangential to the circle $r = \tau\gamma^{-1}$ at the point $\theta = \theta_0$, where $\cos\theta_0 = \gamma^{-1}$. Hence, a leading transverse wave exists for $\theta < \theta_0$. It should be noted that the displacement vector has a singularity of the order $-1/r$ at the fronts of longitudinal ($r = \tau$) and transverse ($r = \tau\gamma^{-1}$) waves. On the transverse wavefront $r = \tau\gamma^{-1}$, following the leading transverse wave, i.e. for $\theta < \theta_0$, this singularity appears as we approach the front from any side. It should be also noted that on the free surface, a singularity exists at the point $x = \tau/\beta$, travelling at the Rayleigh wave velocity. This singularity is of the order -1 and appears only on the free surface. Its presence is due to the existence of a zero at $s = \beta$ in the expression $R(s)$ in the denominators of the functions u and v.

Section 5
The Plane Dynamic Problem of Indentation of a Smooth Punch

The Wiener-Hopf method was applied for the first time for solving dynamic problems of the theory of elasticity (see Subsec. 4, Sec. 1, Ch. 1, Vol. 1), while in-

vestigating the stationary problem of diffraction at a semi-infinite cut with free edges, as well as while studying the state of stress resulting from the instantaneous formation of a semi-infinite interval. These problems have mixed boundary conditions, given on two semi-infinite intervals, for one boundary condition through the entire infinite interval. By way of an example, we shall now solve the plane problem of indentation of a smooth punch [48] to illustrate the application of this method to the dynamic theory of elasticity. For the sake of simplicity, we shall confine ourselves to the case of a semi-infinite punch.

Suppose that in the elastic half-plane $y \geqslant 0$, a smooth punch is pressed at the instant $t = 0$ into a semi-infinite interval with boundary $x \geqslant 0$. The equation of motion of this process is given by $y = f(x, t)$, where $f(x, t)$ is assumed to be a bounded function having a finite number of lines of discontinuity for $x \geqslant 0, t \geqslant 0$. Then, considering that the medium is at rest before the onset of the indentation process, we get the following boundary and initial conditions for determining the displacement vector $\mathbf{u}$:

$$\begin{aligned} &\tau_{xy} = 0 \qquad (-\infty < x < \infty, y = 0), \\ &\sigma_y = 0 \qquad (x < 0, y = 0), \\ &v = f(x, t) \qquad (x \geqslant 0, y = 0), \\ &\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} = 0 \qquad (t = 0). \end{aligned} \tag{5.1}$$

Using the representations (5.50), (5.53), Ch. 3, Vol. 1 for the vector $\mathbf{u}(u, v)$:

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{rot}(\Psi \mathbf{k}), \tag{5.2}$$

we find that the problem is reduced to the solution of the following system of equations for the potentials Φ and Ψ:

$$\Delta\Phi = \frac{\partial^2 \Phi}{\partial \tau^2}, \qquad \Delta\Psi = \gamma^2 \frac{\partial^2 \Psi}{\partial \tau^2} \qquad (\tau = at, \gamma = a/b), \tag{4.2}$$

at the boundary and the initial conditions

$$\tau_{xy} = b^2 \rho \left(2 \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2} \right) = 0 \tag{5.3}$$

$$(y = 0, -\infty < x < \infty),$$

$$\sigma_y = (a^2 - 2b^2)\rho \Delta\Phi + 2b^2 \rho \left(\frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x \partial y} \right) = 0 \qquad (y = 0, x < 0), \tag{5.4}$$

$$v = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x} = f(x, \tau) \qquad (y = 0, x \geqslant 0), \tag{5.5}$$

$$\Phi = \frac{\partial \Phi}{\partial \tau} = \Psi = \frac{\partial \Psi}{\partial \tau} = 0 \qquad (\tau = 0). \tag{5.6}$$

Besides, in order to ensure the uniqueness of the solution of this problem, we must impose certain restriction on the behaviour of the displacement vector in the vicinity of the edge of the punch $x = 0$:

$$\mathbf{u} = C + O(r^{\varepsilon}) \qquad (\varepsilon > 0, r \to 0, \quad C \equiv C(\tau)). \tag{5.7}$$

Thus, the problem has been formulated completely.

Applying the Laplace transformation in τ, followed by a reversible Laplace transformation in x, to Eqs. (4.2) and the boundary conditions (5.3), (5.4), and (5.5); we get the following system of equations and boundary conditions for $\bar{\Phi}^*$ and $\bar{\Psi}^*$:

$$\begin{aligned} \frac{d^2\bar{\Phi}^*}{dy^2} &= (p^2 - s^2)\bar{\Phi}^*, \\ \frac{d^2\bar{\Psi}^*}{dy^2} &= (\gamma^2 p^2 - s^2)\bar{\Psi}^*, \end{aligned} \tag{5.8}$$

$$2s\frac{d\bar{\Phi}^*}{dy} + \frac{d^2\bar{\Phi}^*}{dy^2} - s^2\bar{\Psi}^* = 0 \qquad (y = 0), \tag{5.9}$$

$$\bar{\sigma}_y^+ = (\gamma^2 - 2)p^2\bar{\Phi}^* + 2\left(\frac{d^2\bar{\Phi}^*}{dy^2} - s\frac{d\bar{\Psi}^*}{dy}\right) \qquad (y = 0), \tag{5.10}$$

$$\bar{v}^- + \bar{v}^+ = \frac{d\bar{\Phi}^*}{dy} - s\bar{\Psi}^* \qquad (y = 0). \tag{5.11}$$

The following notation has been used here:

$$\begin{aligned} \bar{\sigma}_y^+ &= \frac{1}{\mu}\int_0^{\infty} e^{-sx}dx \int_0^{\infty} \sigma_y e^{-p\tau}d\tau, \\ \bar{v}^- &= \int_{-\infty}^{0} e^{-sx}dx \int_0^{\infty} v e^{-p\tau}d\tau, \\ \bar{v}^+ &= \int_0^{\infty} e^{-sx}dx \int_0^{\infty} f(x,\tau)e^{-p\tau}d\tau \qquad (\mathrm{Re}\,p > 0), \\ \bar{F} &= \int_0^{\infty} e^{-p\tau}F d\tau, \qquad \bar{F}^* = \int_{-\infty}^{\infty} e^{-sx}\bar{F}dx \qquad (F = \Phi, \Psi); \end{aligned} \tag{5.12}$$

the functions $\bar{\sigma}_y^+$ and $\bar{v}^-$ are unknown.

Let us determine the domain of convergence of the integrals (5.12) in the plane s. Since the perturbations are bounded at least for $x \to +\infty$, and disappear for $x < 0$

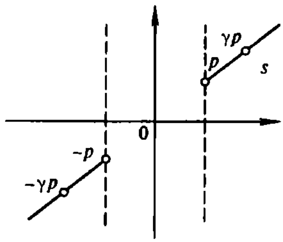

Fig. 55. A system of cuts.

if $\tau + x < 0$, we get

$$\bar{F} = \begin{cases} O(1), & x \to +\infty, \\ O(e^{px}), & x \to -\infty. \end{cases}$$

Hence the integral $\bar{F}^*$ converges and is an analytical function in the interval $0 < \operatorname{Re}s < \operatorname{Re}p$. For this reason, the integrals $\bar{\sigma}_y^+$ and $\bar{v}^-$ converge and are analytical functions in the regions $\operatorname{Re}s > 0$ and $\operatorname{Re}s < \operatorname{Re}p$ respectively. Thus, we find that in formulas (5.8)-(5.12), $0 < \operatorname{Re}s < \operatorname{Re}p$.

Solving Eqs. (5.8) and neglecting solutions which decrease as $y \to \infty$, we get

$$\bar{\Phi}^* = Ae^{-\sqrt{p^2 - s^2}y}, \qquad \bar{\Psi}^* = Be^{-\sqrt{\gamma^2p^2 - s^2}y}. \tag{5.13}$$

Here, cuts have been made in the s-plane from the points $s = \pm p$ and $s = \pm\gamma p$ to infinity along the rays $s = \arg p$ and $s = \pi + \arg p$ in order to isolate the branches of the radicals $\sqrt{p^2 - s^2}$ and $\sqrt{\gamma^2p^2 - s^2}$ (Fig. 55). Besides, it is assumed that the following equalities are satisfied for $s = 0$: $\sqrt{p^2 - s^2} = p$, $\sqrt{\gamma^2p^2 - s^2} = \gamma p$. The following inequalities are then valid:

$$\operatorname{Re}\sqrt{p^2 - s^2} > 0, \qquad \operatorname{Re}\sqrt{\gamma^2p^2 - s^2} > 0 \qquad (0 < \operatorname{Re}s < \operatorname{Re}p).$$

Substituting the expressions (5.13) into the boundary conditions (5.9)-(5.11), we obtain three algebraic equations for determining four unknowns A, B, $\bar{\sigma}_y^+$, and $\bar{v}^-$:

$$\begin{gathered} -2s\sqrt{p^2 - s^2}A + (\gamma^2p^2 - 2s^2)B = 0, \\ (\gamma^2p^2 - 2s^2)A + 2s\sqrt{\gamma^2p^2 - s^2}B = \bar{\sigma}_y^+, \\ -\sqrt{p^2 - s^2}A - sB = \bar{v}^- + \bar{v}^+. \end{gathered}$$

Eliminating A and B, we get the following equation connecting $\bar{\sigma}_y^+$ and $\bar{v}^-$:

$$\frac{\gamma^2p^2\sqrt{p^2 - s^2}}{R(p, s)}\bar{\sigma}_y^+ + \bar{v}^+ + \bar{v}^- = 0 \qquad (0 < \operatorname{Re}s < \operatorname{Re}p), \tag{5.14}$$

$$R(p, s) = (\gamma^2p^2 - 2s^2)^2 + 4s^2\sqrt{(\gamma^2p^2 - s^2)(p^2 - s^2)}.$$

We shall analyze in detail the properties of functions appearing in this equation. With the help of (5.7), we find that the Laplace transforms (in τ) of the functions

σ_y and v for $y = 0$ must have the following asymptotics as $x \to 0$:

$$\begin{aligned} \bar{\sigma}_y &= O(x^\varepsilon) \qquad (\varepsilon > -1, x \to +0), \\ \bar{v} &= O(1) \qquad (x \to -0). \end{aligned} \tag{5.15}$$

Consequently, using the properties of the Laplace transformations, we find that the function σ_y^+, which is analytical in the domain Re$s > 0$, tends to zero as $s \to \infty$ in this domain. The function $\bar{v}^-$, which is analytical in the domain Re$s <$ Rep, decreases at least as s^{-1} for $s \to \infty$ (Re$s <$ Rep).

In order to find the properties of the function $K(p, s) \equiv \gamma^2 p^2 \sqrt{p^2 - s^2}/$ $/R(p, s)$, let us consider the expression for $R(p, s)$. It can be easily seen that

$$R(p,s) = p^4 R\left(\frac{s}{p}\right) = p^4 R(q) \qquad \left(q = \frac{s}{p}\right),$$

where $R(q)$ is the left-hand side of Rayleigh's equation (see. Eq. (7.39), Ch. 5), which is a single-valued analytical function in the plane q with cuts $[-\gamma, -1]$ and $[1, \gamma]$, and has two real zeros $q_{1,2} = \pm\beta\ (\beta > \gamma)$ in it. Besides, the following asymptotic exists:

$$R(q) = -2(\gamma^2 - 1)q^2 + O(1) \qquad (q \to \infty).$$

Hence, the function $K(p, s)$, which is analytical in the interval $|\text{Re}s| <$ Rep, does not have any zeros in it, and decreases as s^{-1} for $s \to \infty$.

With the help of Cauchy's formula, $K(p, s)$ can be represented in the following form in the interval $|\text{Re}s| <$ Rep:

$$K(p,s) = -\frac{\gamma^2\sqrt{1-q^2}}{2p(\gamma^2 - 1)(q^2 - \beta^2)}\, e^{-\ln\left[\frac{R(q)}{2(1-\gamma^2)(q^2-\beta^2)}\right]} = K_+(p,s)K_-(p,s)\frac{\gamma^2}{2p(1-\gamma^2)}, \tag{5.16}$$

$$K_+(p,s) \equiv K_+(q) = \frac{\sqrt{1+q}}{q+\beta}\, e^{\frac{1}{2\pi i}\int_{L_0} \ln\left[\frac{R(z)}{2(1-\gamma^2)(z^2-\beta^2)}\right]\frac{dz}{z-q}},$$

$$K_-(p,s) \equiv K_-(q) = \frac{\sqrt{1-q}}{q-\beta}\, e^{-\frac{1}{2\pi i}\int_{L_1} \ln\left[\frac{R(z)}{2(1-\gamma^2)(z^2-\beta^2)}\right]\frac{dz}{z-q}}.$$

Here, the contours L_0 and L_1 are represented in Fig. 56, while the logarithmic branch has been chosen in such a way that the argument of the logarithm is equal to zero for $z = 0$ (it should be noted that the substitution of q by s/p "shrinks" the s-plane $|p|$ times and rotates it by an angle equal to $-\arg p$). Consequently, the region $|\text{Re}s| <$ Rep (see. Fig. 55) is converted into an inclined strip in the q-plane (see Fig. 56).

The integrand functions in (5.16) are analytical in the plane everywhere outside the cuts $[-\gamma, -1]$ and $[1, \gamma]$, and decrease as $O(z^{-3})$ for $z \to \infty$. Then, deforming the contours L_0 and L_1 along the negative and positive real semi-axes respectively, and assuming that the function $\ln[R(z)\, 2^{-1}(1 + \gamma^2)^{-1}(z^2 - \beta^2)^{-1}]$ assumes complex-conjugate values at the upper $[-\gamma, -1]$ and lower $[1, \gamma]$ edges of the cut,

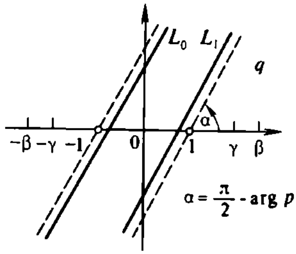

Fig. 56. The L_0 and L_1 contours.

and that

$$\ln(x + iy) = \ln(re^{i\theta}) = \ln r + i\theta = \ln\sqrt{x^2 + y^2} + i \arctan\frac{y}{x},$$

we get the following expressions for $K_+(p, s)$ and $K_-(p, s)$:

$$\begin{aligned} K_+(p,s) &\equiv K_+(q) = \frac{\sqrt{1+q}}{q+\beta}\, e^{-\frac{1}{\pi}\int_{-\gamma}^{-1} g(z)\frac{dz}{z-q}}, \\ K_-(p,s) &\equiv K_-(q) = \frac{\sqrt{1-q}}{q-\beta}\, e^{\frac{1}{\pi}\int_{1}^{\gamma} g(z)\frac{dz}{z-q}}, \\ g(z) &= \arctan\frac{4z^2\sqrt{(\gamma^2 - z^2)(z^2 - 1)}}{(\gamma^2 - 2z^2)^2} \end{aligned} \tag{5.17}$$

(the radical in $g(z)$ is arithmetic).

The function $K_+(q)$ is analytical outside the cut along the real axis $-\infty < q \leqslant -1$ and does not have any zeros there except $q = \infty$. The function $K_-(q)$ is analytical outside the cut along the real axis $1 \leqslant q < \infty$ and does not have any zeros there except $q = \infty$. At infinity, $K_+(q)$ and $K_-(q)$ tend to zero as $q^{-1/2}$. Consequently, $K_+(p, s)$ is analytical in the s-plane outside the cut from the point $s = -p$ to infinity along the ray $\arg s = \arg p + \pi$, while the function $K_-(p, s)$ is analytical outside the cut from the point $s = p$ to infinity along the ray $\arg s = \arg p$. Moreover, the functions $K_+(p, s)$ and $K_-(p, s)$ do not have any zeros at the end points of the plane outside these cuts, and tend to zero as $s^{-1/2}$. Using (5.16), we can rewrite (5.14) in the following form:

$$\frac{\bar{\sigma}_y^+ K_+(q,s)\gamma^2}{2p(\gamma^2 - 1)} - \frac{\bar{v}^+}{K_-(p,s)} = \frac{\bar{v}^-}{K_-(p,s)} \qquad (0 < \mathrm{Re}\, s < \mathrm{Re}\, p). \tag{5.18}$$

Apparently, the second term on the left-hand side of (5.18) is analytical in the interval $0 < \mathrm{Re}\, s < \mathrm{Re}\, p$, and tends to zero as $s^{-1/2}$ for $s \to \infty$.

Hence, this term can be represented in the following form with the help of Cauchy-type integrals:

$$\frac{\bar{v}^+}{K_-(p,s)} = N_+(p,s) + N_-(p,s). \tag{5.19}$$

Here, the following notation has been used:

$$N_+(p,s) = -\frac{1}{2\pi i}\int\limits_{l_1}\frac{\bar{v}^+(z)dz}{K_-(p,z)(z-s)},$$

$$N_-(p,s) = \frac{1}{2\pi i}\int\limits_{l_2}\frac{\bar{v}^+(z)dz}{K_-(p,z)(z-s)}.$$

Here, the contours l_1 and l_2 are straight lines parallel to the imaginary axis, lying inside the region $0 < \mathrm{Re}s < \mathrm{Re}p$ near its left-and right edges respectively.

It can be shown that $N_+(p, s)$ is analytical in the half-plane $\mathrm{Re}s > 0$, and tends to zero at least as s^{-1} for $s \to \infty$ ($\mathrm{Re}s > 0$), while $N_-(p, s)$ is analytical in the domain $\mathrm{Re}s < \mathrm{Re}p$, and tends to zero as $s^{-1/2}$ for $s \to \infty$ ($\mathrm{Re}s < \mathrm{Re}p$).

Equation (5.18) can be rewritten in the following form:

$$\frac{\bar{\sigma}_y^+K_+(p,s)\gamma^2}{2p(\gamma^2-1)} - N_+(p,s) = N_-(p,s) + \frac{\bar{v}^-}{K_-(p,s)} \qquad (0 < \mathrm{Re}s < \mathrm{Re}p). \tag{5.20}$$

Then, taking into account all the above-mentioned properties of the functions $\bar{\sigma}_y^+$, $K_+(p, s)$, $N_+(p, s)$, $K_-(p, s)$, $N_-(p, s)$, and $\bar{v}^-$, we find that the right-hand side of (5.20) is regular for $\mathrm{Re}s > 0$, and decreases more rapidly than $s^{-1/2}$ in this half-plane for $s \to \infty$. The left-hand side is regular in the domain $\mathrm{Re}s < \mathrm{Re}p$, and decreases at least as rapidly as $s^{-1/2}$ in this half-plane for $s \to \infty$. It follows from the equality of these parts in the region $0 < \mathrm{Re}s < \mathrm{Re}p$ that in the complex s-plane, there exists a unique integral function $F(s)$, identical with the right-hand side of Eq. (5.20) in the region $\mathrm{Re}s > 0$, and with its left-hand side in the region $\mathrm{Re}s < \mathrm{Re}p$. Since this function is bounded, it follows from Liouville's theorem [35] that $F(s) \equiv$ const. But since $F(s) \to 0$ for $s \to \infty$, this constant is equal to zero, i.e. $F(s) \equiv 0$. Hence, taking into account the fact that the function $F(s)$ is represented by the right- and left-hand sides of (5.20) in the half-planes $\mathrm{Re}s > 0$ and $\mathrm{Re}s < \mathrm{Re}p$ respectively, we get

$$\frac{\bar{\sigma}_y^+K_+(p,s)}{2p(\gamma^2-1)} - N_+(p,s) = 0 \qquad (\mathrm{Re}s > 0), \tag{5.21}$$

$$N_-(p,s) + \frac{\bar{v}^-}{K_-(p,s)} = 0 \qquad (\mathrm{Re}s < \mathrm{Re}p). \tag{5.22}$$

This leads to the required functions $\bar{\sigma}_y^+$ and $\bar{v}^-$:

$$\bar{\sigma}_y^+ = \frac{2p(\gamma^2-1)}{\gamma^2K_+(p,s)}N_+(p,s) \qquad (\mathrm{Re}s > 0), \tag{5.23}$$

$$\bar{v}^- = -K_-(p,s)N_-(p,s) \qquad (\mathrm{Re}s < \mathrm{Re}p). \tag{5.24}$$

It follows from the properties of $N_\pm(p, s)$ and $K_\pm(p, s)$ that the functions $\bar{\sigma}_y^+$ and $\bar{v}^-$ constructed above satisfy the regularity conditions and the behaviour at infinity formulated for these functions.

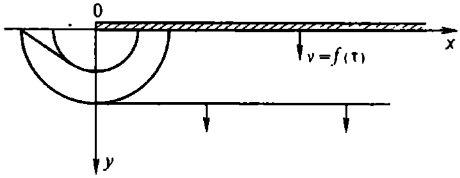

Fig. 57. The wave fronts.

Applying the inverse Laplace transformations in s and p, we can find the original functions:

$$\sigma_y(\tau, x, 0) = \frac{\mu(\gamma^2 - 1)}{2\pi i \gamma^2} \int_{c - i\infty}^{c + i\infty} e^{p\tau} dp \frac{1}{2\pi i} \int_{d - i\infty}^{d + i\infty} \frac{N_+(p, s)}{K_+(p, s)} e^{sx} ds \quad (x > 0), \tag{5.25}$$

$$v(\tau, x, 0) = -\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} e^{p\tau} dp \frac{1}{2\pi i} \int_{d - i\infty}^{d + i\infty} N_-(p, s) K_-(p, s) e^{sx} ds \tag{5.26}$$

$$(x < 0, c > d > 0).$$

Let us consider the solutions (5.25) and (5.26) in greater detail for the specific case when $f(x, \tau) = f(\tau)$ (the wavefront pattern in this case is shown in Fig. 57). Then, $\bar{v}^+ = \bar{f}(p)/s$, and considering that $K_\pm(p, s) = K_\pm(q)$, we get from (5.19)

$$N_+(p, s) = \frac{\bar{f}(p)}{K_-(0)s}, \qquad N_-(p, s) = \frac{\bar{f}(p)}{s}\left[\frac{1}{K_-(q)} - \frac{1}{K_-(0)}\right]. \tag{5.27}$$

Consequently, formulas (5.25) and (5.26) can be transformed as follows:

$$\begin{aligned} \sigma_y(\tau, x, 0) &= \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} p\bar{f}(p)\bar{M}_+(px) e^{p\tau} dp \qquad (x > 0), \\ v(\tau, x, 0) &= \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \bar{f}(p)\bar{M}_-(px) e^{p\tau} dp \qquad (x < 0), \end{aligned} \tag{5.28}$$

where the following notation has been used:

$$M_+(px) = \frac{2\mu(\gamma^2 - 1)}{\gamma^2 K_-(0)} \frac{1}{2\pi i} \int_{d - i\infty}^{d + i\infty} \frac{e^{sx} ds}{K_+(q)s} = \frac{2\mu(\gamma^2 - 1)}{\gamma^2 K_-(0) 2\pi i} \int_{L_1} \frac{e^{pxq} dq}{K_+(q)q} \quad (x > 0), \tag{5.29}$$

$$\bar{M}_-(px) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\left[\frac{K_-(q)}{K_-(0)} - 1\right]\frac{e^{sx}ds}{s}$$

$$= \frac{1}{2\pi i}\int_{L_1}\left[\frac{K_-(q)}{K_-(0)} - 1\right]\frac{e^{pxq}dq}{q} \qquad (x < 0).$$

Deforming the contour L_1 along the cut $[1, +\infty)$ in the expression for $\bar{M}_-(px)$ and along the cut $(-\infty, -1]$ in the expressions for $M_+(px)$ and then replacing q by $-q$ in $M_+(px)$, we can transform Eqs. (5.29) as follows:

$$\frac{\gamma K_-(0)}{2\mu(\gamma^2 - 1)}M_+(px) = \frac{1}{K_+(0)} - \frac{1}{\pi}\int_1^\infty \frac{\beta - q}{\sqrt{q - 1}}e^{-\frac{1}{\pi}\int_1^\gamma g(z)\frac{dz}{z - q} - pxq}$$

$$\times\{1 + [\cos g(q) - 1]H(\gamma - q)\}\frac{dq}{q} \qquad (x > 0), \quad (5.30)$$

$$\bar{M}_-(px) = -\frac{1}{\pi}\int_1^\infty \frac{\sqrt{q - 1}}{q - \beta}e^{\frac{1}{\pi}\int_1^\gamma g(z)\frac{dz}{z - q} + pxq}$$

$$\times\{1 + [\cos g(q) - 1]H(\gamma - q)\}\frac{dq}{q} \qquad (x < 0).$$

Here, by integral from 1 to ∞ in $M_-(px)$, and by integral from 1 to γ (for $1 < q < \gamma$) in $\bar{M}_\pm(px)$, we mean the principal value in the sense of Cauchy; the radicals in (5.30) are arithmetic everywhere. Besides, it should be noted that $K_-(0) = -K_+(0) = -\gamma^{-2}\sqrt{2(\gamma^2 - 1)}$.

Consequently, with the help of (5.30), we get from formulas (5.28)

$$\sigma_y(\tau, x, 0) = -\mu\gamma^2 f'(\tau)$$

$$+ H\left(\frac{\tau}{x} - 1\right)\frac{\mu\sqrt{2(\gamma^2 - 1)}}{\pi}\frac{\partial}{\partial\tau}\int_1^{\tau/x}\frac{\beta - q}{\sqrt{q - 1}}e^{-\frac{1}{\pi}\int_1^\gamma\frac{g(z)dz}{z - q}}$$

$$\times\{1 + [\cos g(q) - 1]H(\gamma - q)\}f(\tau - xq)\frac{dq}{q} \qquad (x > 0), \quad (5.31)$$

$$v(\tau, x, 0) = \frac{\gamma^2 H\left(-1 - \frac{\tau}{x}\right)}{\pi\sqrt{2(\gamma^2 - 1)}}\int_1^{-\tau/x}\frac{\sqrt{q - 1}}{q - \beta}e^{\frac{1}{\pi}\int_1^\gamma\frac{g(z)dz}{z - q}}$$

$$\times\{1 + [\cos g(q) - 1]H(\gamma - q)\}f(\tau + xq)\frac{dq}{q} \qquad (x < 0). \quad (5.32)$$

The derivative $f'(\tau)$ in (5.31) should be taken as a generalized derivative, and in the special case when $f(\tau) = H(\tau)$, we get $f'(\tau) = \delta(t)$.

With the help of the expressions (5.31) and (5.32), let us investigate some properties of the problem of a semi-infinite punch. Let us first consider the expression for

the vertical displacement at the boundary $y = 0, x < 0$. For $\tau + x < 0$, we get $v = 0$, which is in accordance with the zero initial conditions. Here, $\tau + x = 0$ is the equation of the longitudinal wavefront in the free part of the boundary $y = 0$. On this front, $v = 0$, $\dfrac{\partial v}{\partial \tau} = 0$, and $\dfrac{\partial v}{\partial x} = 0$.

On the transverse wavefront (for $y = 0$), $\tau + \gamma x = 0$, in the same way as for the longitudinal wavefront, and $v(\tau, x, 0)$ and both its first derivatives with respect to x and τ are continuous. It can be easily seen from the formula (5.32) that for $\tau = \beta x$, the function $v(\tau, x, 0)$ has a logarithmic singularity. In other words, we have a logarithmic discontinuity in the vertical displacement in the free part of the boundary, which propagates with the Rayleigh wave velocity. It can be also verified that $v(\tau, x, 0)$ is continuous at the point $x = 0$, but $\partial v/\partial x$ turns out to be infinite for $x \to -0$, and has an asymptote for small values of x:

$$\frac{\partial v}{\partial x} \sim M(\tau)(-x)^{-1/2}.$$

From the formula (5.31), we get the following expression for the stress $\sigma_y(\tau, x, 0)$ under the punch ($x > 0$):

$$\sigma_y(\tau, x, 0) = -\mu\gamma^2 f'(\tau) \qquad \text{for} \qquad \tau - x < 0.$$

In other words, the stress at points where perturbation has not reached from the edge $x = 0$ is the same as it would be if the condition $\tau_{xy} = 0, v = f'(\tau)$ were imposed for the entire boundary. This is what should be expected, since the dynamic equations of the theory of elasticity are hyperbolic in nature. At the moment $\tau = x$ the longitudinal wavefront arrives at the point x under the punch. On this front, $\sigma_y = 0$ if $f(0) = 0$ (here, $f(0)$ denotes $\lim\limits_{\tau \to +0} f(\tau)$, and $\sigma_y \to \infty$ as $\varphi(x)/\sqrt{\tau - x}$ if $f(0) \neq 0$. Moreover, when $x \to +\infty$, $\varphi(x)$ tends to zero as x^{-1}. In the expression (5.31) for $\sigma_y(\tau, x, 0)$, we can also isolate the wavefront of the propagating transverse wave $\tau - \gamma x = 0$. In this case, such conditions exist under the punch ($x > 0$) that upon interaction with the boundary, a propagating longitudinal or transverse wave generates only a longitudinal or a transverse wave respectively. Thus, the stress σ_y is the stress in the longitudinal wave for $\tau/\gamma < x < \tau$ (see Fig. 57), and the total stress in the longitudinal and transverse waves for $0 < x < \tau$. As in the case of the corresponding static problem, the following integrable singularity of the type $x^{-1/2}$ takes place at the point $x = 0$:

$$\sigma_y = -\frac{\mu\sqrt{2(\gamma^2 - 1)}}{\pi\sqrt{x}} \frac{\partial}{\partial \tau} \int_0^\tau \frac{f(q)\,dq}{\sqrt{\tau - q}} \qquad (x \to 0).$$

Before concluding this section, it should be noted that for a punch of finite width ($0 < x < l$), the solution may be obtained by superimposing the solutions for semi-infinite punches. This result is based on the fact that the equations of the dynamic theory of elasticity are hyperbolic in nature and hence the perturbations propagate with a finite velocity. Consequently, the solution for a semi-infinite punch remains valid until the diffracted waves from the opposite edge reach the region under investigation.

Section 6
The Problem on the Growth of a Crack at a Varying Velocity

From the mathematical point of view, the plane problems on the dynamic growth of a crack with a varying velocity can be reduced to the solution of the hyperbolic system of equations (4.2) with mixed boundary conditions given on the plane (one of these conditions is continuous across the plane) for the case when the boundary between the regions for which the mixed boundary conditions are given moves with a varying velocity.

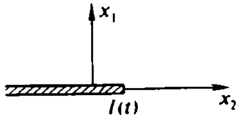

Fig. 58. A semi-infinite cut in a space.

Following [49], we shall give the basic results of this problem.

Suppose that an infinite elastic medium can be represented as a space with a semi-infinite cut (see Fig. 58)

$$x_1 = 0, \quad -\infty < x_2 < l(t), \quad -\infty < x_3 < \infty. \tag{6.1}$$

We shall assume that all the conditions imposed on the medium are independent of the coordinate x_3 (plane problem). Taking this into consideration, we can rewrite the equations of motion (1.11), Ch. 2, Vol. 1 and introduce somewhat different notation:

$$\sigma_{\alpha\beta,\beta} = \rho\ddot{u}_\alpha, \qquad \sigma_{3\alpha,\alpha} = \rho\ddot{u}_3 \qquad (\alpha, \beta = 1, 2). \tag{6.2}$$

Then, Hooke's law can be rewritten in the form

$$\sigma_{\alpha\beta} = \mu[\delta_{\alpha\beta}(\gamma^2 - 2)u_{\lambda,\lambda} + u_{\alpha,\beta} + u_{\beta,\alpha}] \qquad (\alpha, \beta, \lambda = 1, 2), \tag{6.3}$$

$$\sigma_{3\alpha} = \mu u_{3,\alpha} \qquad \left(\gamma = \frac{a}{b}\right).$$

In Eqs. (6.2) the dots indicate differentiation with respect to time t, while the index after a comma means differentiation with respect to the corresponding coordinate. Here and henceforth, the Greek indices assume the values 1 and 2, while the Latin ones assume the values 1, 2, and 3 (for a recurring Greek index, summation is carried out). As usual, we shall seek a solution of the equations of motion (6.2) in the absence of mass forces, which satisfies the homogeneous initial conditions

$$u_k = \dot{u}_k = 0 \quad \text{for} \quad t \leqslant 0. \tag{6.4}$$

Suppose that the stresses $-p_i(x_2, t)$ are specified at the cut. Then, the boundary conditions can be written in the following form:

$$\sigma_i(x_2, t) \equiv \sigma_{i1}(0, x_2, t) = -p_i(x_2, t) \qquad \text{for} \qquad x_1 = 0, \ x_2 < l(t). \tag{6.5}$$

For the function $l(t)$ which defines the law of growth of the crack, we assume that

$$0 \leqslant \dot{l}(t) < b. \tag{6.6}$$

Equations (6.2) and (6.3) will be solved with the help of the Laplace double transformation in time t and in coordinate x_2. Here, the transforms of the functions will be denoted by the same letters as were used for the original functions, and the distinction, where necessary, will be made by writing the arguments explicitly:

$$u_i(x_1, q, p) = \int_{-\infty}^{\infty} e^{-qx_2} dx_2 \int_0^{\infty} u_i(x_1, x_2, t) e^{-pt} dt$$
$$(0 < Re\, q < Re\, p), \tag{6.7}$$
$$\sigma_{ik}(x_1, q, p) = \int_{-\infty}^{\infty} e^{-qt} dx_2 \int_0^{\infty} \sigma_{ik}(x_1, x_2, t) e^{-pt} dt.$$

For the sake of brevity, we shall denote by $\sigma_i(q, p)$ the Laplace transformation of the boundary values of the components of the stress vector on the x_2-axis: $\sigma_i(x_2, t) \equiv \sigma_{i1}(0, x_2, t)$. Solving Eqs. (6.2) and (6.3), it is convenient to express the required displacements in terms of stresses on the plane $x_1 = 0$, since in the problem under consideration the stresses are continuous upon passing through this plane. For this purpose, we shall solve Eqs. (6.2) and (6.3) as follows. In accordance with the results of Sec. 5, Ch. 3, Vol. 1, a solution of Eqs. (6.2) and (6.3) for the displacement vector $\mathbf{u}(u_1, u_2, u_3)$, which is independent of x_3, can be reduced to a solution of the wave equations (5.51), (5.52), Ch. 3, Vol. 1 for determining the potentials Φ, Ψ_1, and Ψ_2, which are related to the vector $\mathbf{u}$ through the following formula, obtained with the help of Eqs. (5.50), (5.57), Ch. 3, Vol. 1:

$$\mathbf{u} = \operatorname{grad}\Phi + \operatorname{curl}(\Psi_1\mathbf{k}) + \operatorname{curl}\operatorname{curl}(\Psi_2\mathbf{k}). \tag{6.8}$$

Applying the Laplace double transformation in x_2 and t to the wave equations and solving the equations thus obtained for the transformations $\Phi(x_1, q, p)$, $\Psi_1(x_1, q, p)$, and $\Psi_2(x_1, q, p)$ in the half-space $x_1 > 0$, we find

$$\Phi(x_1, q, p) = A e^{-x_1\sqrt{a^{-2}p^2 - q^2}},$$
$$\Psi_j(x_1, q, p) = B_j e^{-x_1\sqrt{b^{-2}p^2 - q^2}} \qquad (j = 1, 2). \tag{6.9}$$

Further, assuming that the stress components $\sigma_{i1}(0, x_2, t)$ are given in the plane $x_1 = 0$, we get the following boundary conditions at $x_1 = 0$ for determining $\mathbf{u}$:

$$\sigma_{\alpha 1}(0, x_2, t) = \mu[\delta_{\alpha 1}(\gamma^2 - 2)u_{\lambda,\lambda} + u_{\alpha,1} + u_{1,\alpha}] \qquad (\alpha = 1, 2),$$
$$\sigma_{31} = \mu u_{3,1}. \tag{6.10}$$

The constants A, B_1, and B_2 in (6.9) can be uniquely determined from the three conditions (6.10). Thus, the solution for the function $\mathbf{u}(u_1, u_2, u_3)$ in the half-space $x_1 > 0$ is expressed in terms of stresses on the plane $x_1 = 0$.

Similarly, solving wave equations in the region $x_1 < 0$, we get

$$\Phi(x_1, q, p) = A^0 e^{x_1\sqrt{a^{-2}p^2 - q^2}},$$
$$\Psi_j(x_1, q, p) = B_j^0 e^{x_1\sqrt{b^{-2}p^2 - q^2}} \qquad (j = 1, 2). \tag{6.11}$$

The constants A^0, B_1^0, and B_2^0 are also determined from the boundary conditions (6.10), and hence the expression for the vector $\mathbf{u}(u_1, u_2, u_3)$ in the half-space $x_1 < 0$ is obtained in terms of stresses on the plane $x_1 = 0$.

Consequently, the expressions obtained for $u_i(x_1, q, p)$ are given by the following formulas, which are applicable for $x_1 > 0$ and $x_1 < 0$:

$$
\begin{aligned}
u_1(x_1, q, p) = \frac{1}{\mu p R(q/p)} \Bigg\{ & e^{-|x_1|\sqrt{a^{-2}p^2 - q^2}} \\
& \times \left[2\frac{q}{p} \sqrt{a^{-2} - \left(\frac{q}{p}\right)^2} \sqrt{b^{-2} - \left(\frac{q}{p}\right)^2} \sigma_2(q, p) \right. \\
& \left. - \left(b^{-2} - 2\left(\frac{q}{p}\right)^2\right) \sqrt{a^{-2} - \left(\frac{q}{p}\right)^2} \sigma_1(q, p) \operatorname{sign} x_1 \right] \\
& - \frac{q}{p} e^{-|x_1|\sqrt{b^{-2}p^2 - q^2}} \left[\left(b^{-2} - 2\left(\frac{q}{p}\right)^2\right) \sigma_2(q, p) \right. \\
& \left. + \frac{2q}{p} \sqrt{a^{-2} - \left(\frac{q}{p}\right)^2} \sigma_1(q, p) \operatorname{sign} x_1 \right] \Bigg\},
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
& u_2(x_1, q, p) \\
& = \frac{1}{\mu p R(q/p)} \Bigg\{ \frac{q}{p} e^{-|x_1|\sqrt{a^{-2}p^2 - q^2}} \left[\left(b^{-2} - 2\left(\frac{q}{p}\right)^2\right) \sigma_1(q, p) \right. \\
& \left. - 2\frac{q}{p} \sqrt{b^{-2} - \left(\frac{q}{p}\right)^2} \sigma_2(q, p) \operatorname{sign} x_1 \right] - e^{-|x_1|\sqrt{b^{-2}p^2 - q^2}} \\
& \times \left[2\frac{q}{p} \sqrt{a^{-2} - \left(\frac{q}{p}\right)^2} \sqrt{b^{-2} - \left(\frac{q}{p}\right)^2} \sigma_1(q, p) \right. \\
& \left. - \left(b^{-2} - 2\left(\frac{q}{p}\right)^2\right) \sqrt{b^{-2} - \left(\frac{q}{p}\right)^2} \sigma_2(q, p) \operatorname{sign} x_1 \right] \Bigg\},
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
& u_3(x_1, q, p) \\
& = - \frac{1}{\mu p \sqrt{b^{-2} - \left(\frac{q}{p}\right)^2}} e^{-|x_1|\sqrt{b^{-2}p^2 - q^2}} \sigma_3(q, p) \operatorname{sign} x_1,
\end{aligned}
\tag{6.14}
$$

where

$$
R(s) = (2s^2 - b^{-2})^2 + 4s^2\sqrt{a^{-2} - s^2}\sqrt{b^{-2} - s^2},
$$

$$
\operatorname{sign} x_1 = \begin{cases} 1, & x_1 > 0, \\ -1, & x_1 < 0. \end{cases}
$$

We denote the displacement jump on the x_2-axis by $w_i(x_2, t)$: $w_i(x_2, t) = u_i(+0, x_2, t) - u_i(-0, x_2, t)$, and the corresponding Laplace transform by $w_i(q, p)$. Then, from (6.12)-(6.14), we get

$$\sigma_i(q,p) + pT_i\left(\frac{q}{p}\right) w_i(q,p) = 0, \tag{6.15}$$

where $T_i(s)$ are given by the expressions

$$T_1(s) = \frac{\mu b^2 R(s)}{2\sqrt{a^{-2} - s^2}}, T_2(s) = \frac{\mu b^2 R(s)}{2\sqrt{b^{-2} - s^2}}, T_3(s) = \frac{\mu}{2}\sqrt{b^{-2} - s^2}. \tag{6.16}$$

The expressions (6.12)-(6.14) would give the solution of the problem if the functions $\sigma_i(q, p)$ could be calculated from the boundary conditions, i.e. if the stresses were known over the entire x_2-axis. However, the conditions (6.5) give the values of stresses only at the cut. On the other hand, the displacements must be continuous outside the crack, i.e.

$$w_i(x_2, t) = 0 \quad \text{for} \quad x_2 > l(t). \tag{6.17}$$

Thus, the problem has been reduced to finding stresses $\sigma_i(x_2, t)$ at the continuation of the crack from the conditions (6.5) and (6.17). The transform $\sigma_i(q, p)$ must satisfy the functional equation (6.15).

It can be shown that the following representation is valid for the function $R(s)$:

$$R(s) = (b^{-2} - a^{-2})(c^{-2} - s^2)S(s)S(-s), \tag{6.18}$$

where

$$S(s) = \exp\left[-\frac{1}{\pi}\int_{a^{-2}}^{b^{-2}} \arctan\frac{4\xi^2\sqrt{(\xi^2 - a^{-2})(b^{-2} - \xi^2)}}{(2\xi^2 - b^{-2})^2}\frac{d\xi}{\xi + s}\right] \tag{6.19}$$

and $s = \pm c^{-1}$ are the roots of Rayleigh's equation (c is the velocity of propagation of a Rayleigh wave).

The function $S(s)$ is regular, does not vanish in the complex plane s cut along a segment of the real axis from the point $s = -a^{-1}$ to $s = -b^{-1}$, and tends to unity as $s \to \infty$. The expansion (6.18) is the basis for the solution of Eq. (6.15) obtained by the Wiener-Hopf technique for a semi-infinite crack which is either stationary, or moves at a constant velocity.

Let us now consider a crack which propagates with a sub-Rayleigh velocity ($\dot{l} < c$). For the sake of convenience, we shall henceforth omit the index of the coordinate x_2. It is necessary to find such a solution of Eq. (6.15) under the conditions (6.5) and (6.17) that $\sigma_i(x, t)$ tend to infinity as $[x - l(t)]^{-1/2}$ when $x \to l(t)$:

$$\sigma_i(x, t) = \frac{K_i(t)}{\sqrt{2[x - l(t)]}} + O(1) \text{ for } x \to l(t) + 0. \tag{6.20}$$

We introduce new functions $F_i(x, t)$ and $G_i(x, t)$ which are defined as follows:

$$F_\alpha(q,p) = \frac{\sqrt{a^{-1} + q/p}\,\sqrt{b^{-1} - q/p}}{c^{-1} + q/p} S^{-1}\left(\frac{q}{p}\right)\sigma_\alpha(q,p) \qquad (\alpha = 1,2),$$

$$F_3(q,p) = \sigma_3(q,p); \tag{6.21}$$

$$G_\alpha(q,p) = \frac{\mu(1-\gamma^{-2})(c^{-1} - q/p)}{4\sqrt{a^{-1} - q/p}\,\sqrt{b^{-1} - q/p}} S\left(-\frac{q}{p}\right) w_\alpha(q,p)\,(\alpha = 1,2),$$

$$G_3(q,p) = \frac{1}{2}\mu w_3(q,p). \tag{6.22}$$

Equation (6.15) then assumes the form

$$\frac{1}{p\sqrt{a_i^{-2} - \left(\frac{q}{p}\right)^2}} F_i(q,p) + G_i(q,p) = 0, \tag{6.23}$$

where $a_1 = a_3 = b$ and $a_2 = a$. Applying the inverse Laplace transformation

$$f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{qx}dq\,\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{pt}f(q,p)dp\,(-a < c < 0),$$

we find that the relation (6.23) in the physical variables x and t assumes the form

$$F_i(x,t) = \sigma_i(x,t) - (1-\delta_{i3})\frac{\partial A}{\partial t}, \tag{6.24}$$

$$A = D(-c^{-1})\sqrt{c^{-1} - a^{-1}}\sqrt{c^{-1} - b^{-1}}\int_0^{ct}\sigma_i(x-\eta, t - c^{-1}\eta)d\eta$$

$$+ \frac{1}{2\pi}\int_{a^{-1}}^{b^{-1}}\{D(-s)\}\frac{\sqrt{s - a^{-1}}\sqrt{b^{-1} - s}}{c^{-1} - s}\int_0^{t/s}\sigma_i(x-\eta, t - s\eta)d\eta\,ds,$$

where the following notation has been used:

$$D(s) = [S(s)]^{-1}, \qquad \delta_{i3} = \begin{cases} 0, & i \neq 3, \\ 1, & i = 3, \end{cases}$$

and the braces indicate the sum from above and below of the limiting values of the functions on the real axis:

$$\{D(s)\} = D(s + i0) + D(s - i0).$$

In an exactly similar way, we find from (6.22)

$$G_i(x, t) = c_i \left[w_i(x, t) + (1 - \delta_{i3}) \times \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{a^{-1}}^{b^{-1}} \frac{(c^{-1} - s)\{S(s)\}}{\sqrt{s - a^{-1}}\sqrt{b^{-1} - s}} \int_0^{t/s} w_i(x + \eta, t - s\eta) d\eta \, ds \right], \tag{6.25}$$

where

$$c_\alpha = \frac{1}{4}\mu(1 - \gamma^{-2}) \qquad (\alpha = 1, 2), \qquad c_3 = \frac{1}{2}\mu.$$

The transformations (6.24) and (6.25) have the property that for $x < l(t)$ (i.e. at the crack), the functions $F_i(x, t)$ are calculated in terms of the values of $\sigma_i(x', t')$, where $x' < l(t')$ (it should be recalled that $\dot{l}(t) < c$ by assumption). In the same way, we can calculate the value of $G_i(x, t)$ on the extension of the cut from the values of w_i on the extension of the cut in accordance with (6.25). Hence, instead of the conditions (6.15) and (6.17), we get

$$\begin{aligned} F_i(x, t) &= -f_i(x, t) \qquad \text{for} \qquad x < l(t), \\ G_i(x, t) &= 0 \qquad \text{for} \qquad x > l(t), \end{aligned} \tag{6.26}$$

where $f_i(x, t)$ is a known function connected to $p_i(x, t)$ through the transformation (6.24):

$$f_i(x, t) = p_i(x, t) - (1 - \delta_{i3}) \frac{\partial B}{\partial t}, \tag{6.27}$$

$$B = D(-c^{-1})\sqrt{c^{-1} - a^{-1}}\sqrt{c^{-1} - b^{-1}} \int_0^{ct} p_i(x - \eta, t - c^{-1}\eta) d\eta$$
$$+ \frac{1}{2\pi} \int_{a^{-1}}^{b^{-1}} \{D(-s)\} \frac{\sqrt{s - a^{-1}}\sqrt{b^{-1} - s}}{c^{-1} - s} \int_0^{t/s} p_i(x - \eta, t - s\eta) d\eta \, ds.$$

The transformations of the type of convolution (6.24) and (6.25) are reversible. The inverse transformations have the form

$$\sigma_i(x, t) = F_i(x, t) + (1 - \delta_{i3}) \times \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{a^{-1}}^{b^{-1}} [S(-s)] \frac{c^{-1} - s}{\sqrt{(s - a^{-1})(b^{-1} - s)}} \int_0^{t/s} F_i(x - \eta, t - s\eta) d\eta \, ds, \tag{6.28}$$

$$w_i(x, t) = c_i^{-1}\Bigg[G_i(x, t) - (1 - \delta_{i3})\frac{\partial}{\partial t}\Bigg(D(-c^{-1})$$

$$\times\sqrt{c^{-1} - a^{-1}}\sqrt{c^{-1} - b^{-1}}\int_0^{ct} G_i(x + \eta, t - c^{-1}\eta)d\eta$$

$$+ \frac{1}{2\pi}\int_{a^{-1}}^{b^{-1}} \{D(-s)\}\frac{\sqrt{1 - a^{-1}}\sqrt{b^{-1} - s}}{c^{-1} - s}\int_0^{c/s} G_i(x + \eta, t - s\eta)d\eta\, ds\Bigg)\Bigg]. \tag{6.29}$$

Thus the problem has been reduced to finding the functions $F_i(x, t)$ from Eq. (6.23) and the boundary conditions (6.26).

The asymptotic behaviour of $F_i(x, t)$ for $x \to l(t)$ is similar to the behaviour of $\sigma_i(x, t)$:

$$F_i(x, t) = \frac{m_i(t)}{\sqrt{2[x - l(t)]}} \qquad \text{for} \qquad x \to l(t) + 0, \tag{6.30}$$

where the stress intensity factors $m_i(t)$ are calculated in terms of $k_i(t)$ with the help of Eq. (6.24) as follows:

$$m_i(t) = k_i(t)\left[\delta_{i3} + (1 - \delta_{i3})D\left(-\frac{1}{v(t)}\right)\frac{\sqrt{1 - \dfrac{v(t)}{a}}\sqrt{1 - \dfrac{v(t)}{b}}}{1 - \dfrac{v(t)}{c}}\right]. \tag{6.31}$$

Here, $v(t) = \dot{l}(t)$ is the velocity of propagation of the crack at the instant of time t.

Applying Laplace's inverse transformation to Eq. (6.23), we get

$$\frac{1}{\pi}\iint_{\Delta_i} \frac{F_i(x, t)dx\,dt}{\sqrt{(t_0 - t)^2 a_i^2 - (x_0 - x)^2}} = -\frac{1}{a^2}G(x_0, t_0), \tag{6.32}$$

where Δ_i is a triangle

$$a_i^2(t_0 - t)^2 - (x_0 - x)^2 \geqslant 0 \qquad (0 \leqslant t \leqslant t_0). \tag{6.33}$$

In particular, for $x_0 > l(t_0)$, we get the following relation in view of (6.26):

$$\frac{1}{\pi}\iint_{\Delta_i} \frac{F_i(x, t)dx\,dt}{\sqrt{a_i^2(t_0 - t)^2 - (x_0 - x)^2}} = 0. \tag{6.34}$$

Solutions of equations of the type (6.34) are considered in the theory of supersonic flow past a thin airfoil [50]. Since the edge of the crack does not lie in the region Δ_i for $x_0 > a_i t_0 + l(0)$, it can be easily shown that $F_i(x, t) \equiv 0$ for $x > a_i t + l(0)$. Next, we introduce characteristic variables

$$\xi = (a_i t - x)\sqrt{2}, \qquad \eta = (a_i t + x)\sqrt{2} \tag{6.35}$$

and denote the coordinate of the edge of the crack in the characteristic variables by $\eta^*(\xi)$. This coordinate is determined from the solution of the equation

$$\eta^* - \xi = \sqrt{2}\, l\left(\frac{\eta^* + \xi}{a_i\sqrt{2}}\right). \tag{6.36}$$

Equation (6.34) can then be written in the form

$$\frac{1}{\pi}\int\limits_{-l(0)/\sqrt{2}}^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}} \int\limits_{-\xi}^{\eta_0} \frac{F_i(\xi, \eta)d\eta}{\sqrt{\eta_0 - \eta}} = 0 \qquad \text{for} \quad \eta_0 > \eta^*(\xi_0). \tag{6.37}$$

This is nothing but the Abel equation (2.37), Ch. 1, Vol. 1 in the function

$$\int\limits_{-\xi}^{\eta_0} \frac{F_i(\xi, \eta)d\eta}{\sqrt{\eta_0 - \eta}}.$$

Inverting the Abel operator with respect to ξ, we get

$$\int\limits_{-\xi}^{\eta_0} \frac{F_i(\xi, \eta)d\eta}{\sqrt{\eta_0 - \eta}} = 0. \tag{6.38}$$

Since $F_i(\xi, \eta)$ are known in view of (6.26) for $\eta < \eta^*(\xi)$, this equation can be rewritten in the form

$$\int\limits_{\eta^*(\xi)}^{\eta_0} \frac{F_i(\xi, \eta)d\eta}{\sqrt{\eta_0 - \eta}} = \int\limits_{-\xi}^{\eta^*(\xi)} \frac{f_i(\xi, \eta)}{\sqrt{\eta_0 - \eta}}\, d\eta. \tag{6.39}$$

The solution of this Abel's integral equation can be given in the form

$$F_i(\xi_0, \eta_0) = \frac{1}{\pi\sqrt{\eta_0 - \eta^*(\xi_0)}} \int\limits_{-\xi_0}^{\eta^*(\xi_0)} f_i(\xi_0, \eta)\, \frac{\sqrt{\eta^*(\xi_0) - \eta}}{\eta_0 - \eta}\, d\eta. \tag{6.40}$$

In terms of the physical variables x and t, this solution can be written as follows:

$$F_i(x_0, t_0) = \frac{1}{\pi\sqrt{x_0 - l(t^*)}} \times \int\limits_{x_0 - a_i t_0}^{l(t_i^*)} f_i\left(x, t_0 - \frac{x_0 - x}{a_i}\right) \frac{\sqrt{l(t^*) - x}}{x_0 - x}\, dx, \tag{6.41}$$

where t_i^* is the solution of the equation

$$a_i t_0 - x_0 = a_i t_i^* - l(t_i^*). \tag{6.42}$$

Equations (6.41), (6.27), and (6.28) can now be used to get a solution of the problem for a semi-infinite crack. The uniqueness of the solution of this problem

directly follows from the solution of the corresponding homogeneous problem (i.e. when we put $p_i(x, t) \equiv 0$ in (6.15)).

In particular, we get the following expression from (6.41) for the stress intensity factor:

$$m_i(t) = \frac{\sqrt{2\left[1 - \frac{v(t_0)}{a_i}\right]}}{\pi} \times \int_{l(t_0) - a_i t_0}^{l(t_0)} f_i\left(x, t_0 - \frac{l(t_0) - x}{a_i}\right) \frac{dx}{\sqrt{l(t_0) - x}} \tag{6.43}$$

or, in view of (6.31),

$$k_i(t) = \frac{\sqrt{2}}{\pi}\left[\delta_{i3}\sqrt{1 - \frac{v(t)}{b}} + (1 - \delta_{i3})S\left(-\frac{1}{v(t)}\right) \times \frac{\left(1 - \frac{v(t)}{c}\right)\sqrt{1 - \frac{v(t)}{a_i}}}{\sqrt{1 - \frac{v(t)}{a}}\sqrt{1 - \frac{v(t)}{b}}} \int_0^{a_i t_0} f_i\left(l(t) - x, t - \frac{x}{a_i}\right)\frac{dx}{\sqrt{x}}\right]. \tag{6.44}$$

This expression can be rewritten in the following form:

$$k_i(t) = K_i(v(t))k_{i0}(l(t), t),$$

where we have used the notation

$$k_i(t) = \delta_{i3}\sqrt{1 - \frac{v}{b}} + (1 - \delta_{i3})S\left(-\frac{1}{v}\right)\frac{\left(1 - \frac{v}{b}\right)\sqrt{1 - \frac{v}{a_i}}}{\sqrt{1 - \frac{v}{a}}\sqrt{1 - \frac{v}{b}}}, \tag{6.45}$$

$$k_{i0}(l, t) = \frac{\sqrt{2}}{\pi}\int_0^{a_i t} f_i\left(l - x, t - \frac{x}{a_i}\right)\frac{dx}{\sqrt{x}}. \tag{6.46}$$

Equation (6.46) contains l as a parameter. In other words, this expression will have the same value for any law of propagation of the crack till the instant t, including the particular case when the edge of the crack is situated from the very beginning at the point l. On the other hand, $v = 0$ in this case, and the expression (6.45) becomes unity. It then follows from (6.44) that $k_{i0}(l, t)$ is the stress intensity factor of a stationary crack with its edge at the point l and subjected to the same load. Moreover, it should be noted that for $i = 3$, the solution obtained above remains valid for $c < \dot{l}(t) < b$ as well.

For the case of a crack of finite dimensions, which suddenly begins to grow from a certain static position on both sides with varying velocities, the above solution remains valid until the perturbations from opposite edges start interacting. By a successive superposition of solutions for semi-infinite cracks, we can obtain, in principle, a solution for a finite cut at any finite interval of time.

Let us consider in detail the problem for a longitudinal shear crack, for which the formulas obtained above are considerably simplified. In this case, $i = 3$ and from the formulas (6.41), (6.27), and (6.28), we get

$$\sigma_3(x_0, t_0) = \frac{1}{\pi\sqrt{x_0 - l(t^*)}}$$

$$\times \int\limits_{x_0 - bt}^{l(t_3^*)} P_3\left(x, t_0 - \frac{x_0 - x}{b}\right) \frac{\sqrt{l(t^*) - x}}{x_0 - x}\, dx,$$

where t_3^* is a solution of the equation $bt_0 - x_0 = bt_3^* - l(t_3^*)$, and the following expression holds for the stress intensity factor:

$$k_3(t) = \sqrt{1 - \frac{v}{b}}\, k_{30}(l, t), \tag{6.47}$$

where

$$k_{30}(l, t) = \frac{\sqrt{2}}{\pi} \int\limits_0^{bt} f_3\left(l - x, t - \frac{x}{b}\right) \frac{dx}{\sqrt{x}}\,.$$

Griffith's energy criterion for a linearly elastic brittle medium for the dynamic growth of a longitudinal shear crack can be written in the following form with the help of the intensity factor k_3 [51]:

$$2P(v) = \frac{\pi}{2\mu v} \frac{k_3^2}{\sqrt{v^{-2} - b^{-2}}}, \tag{6.48}$$

where k_3 is given by the formula (6.47), and $P(v)$ is the effective surface energy, which is considered for the given material as the characteristic function for the velocity of propagation of the crack. Its value is determined experimentally or theoretically with the help of certain assumptions about the fracture mechanism. Knowing $P(v)$, we can determine the law of growth of the crack with the help of Eq. (6.48).

Section 7
Solution of Dynamic Problems for a Wedge under Mixed Boundary Conditions

We shall now describe a method for constructing exact analytical solutions of three-dimensional dynamic problems in the theory of elasticity for a wedge under

mixed[2] boundary conditions [52]. This method includes integral transformations as well as the separation of singularities of transforms of the required functions in the vicinity of the edge.

To illustrate this method, let us consider, by way of an example, the solution of the three-dimensional problem of diffraction of a longitudinal wave at a smooth rigid wedge.

Consider an elastic medium filling the domain $r > 0$, $0 < \varphi < \pi/l$, $-\infty < z < \infty$ and bordering the wedge $(\pi/l < \varphi < 2\pi)$ at whose faces $\varphi = 0$, π/l the conditions $w_\varphi = 0$, $\sigma_{\varphi r} = \sigma_{\varphi z} = 0$ are specified (here, r, φ, z are cylindrical coordinates, and the z-axis coincides with the edge of the wedge). At the instant $\tau = -r_0$ $(\tau = at,\ r_0 > 0)$, a source of a spherical longitudinal wave begins to act at the point $(r_0, \varphi_0, 0)$. The potential of this wave is given by

$$\Phi_0 = \frac{1}{R} f(\tau + r_0 - R),$$
$$R = [z^2 + r^2 + r_0^2 - 2rr_0\cos(\varphi - \varphi_0)]^{1/2}, \tag{7.1}$$

and its wavefront arrives at the surface of the wedge at the instant t_*. Here, $-r_0 < \tau_* \leqslant 0$, and $f(\tau)$ is an arbitrary function satisfying the condition of applicability of the Laplace transformation. Moreover, for $\tau < 0$, $f(\tau) \equiv 0$. The wedge introduces a perturbation $\mathbf{u} \equiv \{u_r, u_\varphi, u_z\}$ into the field of displacements of the incident wave. Taking into account the transition to cylindrical coordinates, this perturbation can be described in terms of the longitudinal potential Φ and two scalar transverse potentials Ψ_1 and Ψ_2, given by formulas (5.50) and (5.57), Ch. 3, Vol. 1:

$$u_r = \frac{\partial\Phi}{\partial r} + \frac{\partial\Psi_1}{r\partial\varphi} + \frac{\partial^2\Psi_2}{\partial r\partial z},$$
$$u_\varphi = \frac{\partial\Phi}{r\partial\varphi} - \frac{\partial\Psi_1}{\partial r} + \frac{1}{r}\frac{\partial^2\Psi_2}{\partial\varphi\partial z}, \tag{7.2}$$
$$u_z = \frac{\partial\Phi}{\partial z} - \frac{1}{r^2}\frac{\partial^2\Psi_2}{\partial\varphi^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi_2}{\partial r}\right).$$

It can be easily verified that the boundary conditions on the wedge will be satisfied when the following equalities hold:

$$\frac{\partial\Phi}{\partial\varphi} = -\frac{\partial\Phi_0}{\partial\varphi}, \quad \Psi_1 = 0, \quad \frac{\partial\Psi_2}{\partial\varphi} = 0 \quad \left(\varphi = 0,\ \frac{\pi}{l}\right).$$

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[2] By mixed conditions, we mean the case when normal displacement and shearing stress and, conversely, normal stress and tangential displacement are given on the boundary half-planes.

Consequently, considering the fact that perturbations do not appear until the instant $\tau = \tau_*$, we get the following three systems of equations, boundary conditions, and initial conditions for determining Φ, Ψ_1, and Ψ_2:

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial\tau^2} \qquad \left(\Delta \equiv \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + r^{-2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right),$$
$$\frac{\partial\Phi}{\partial\varphi} = -\frac{\partial\Phi_0}{\partial\varphi} \quad \left(\varphi = 0, \frac{\pi}{l}\right), \qquad \Phi = 0 \quad (\tau < \tau_*); \tag{7.3}$$

$$\Delta\Psi_1 = \gamma^2\frac{\partial^2\Psi_1}{\partial\tau^2} \qquad \left(\gamma = \frac{a}{b} > 1\right),$$
$$\Psi_1 = 0 \quad \left(\varphi = 0, \frac{\pi}{l}\right), \qquad \Psi_1 = 0 \quad (\tau < \tau_*); \tag{7.4}$$

$$\Delta\Psi_2 = \gamma^2\frac{\partial^2\Psi_2}{\partial\tau^2},$$
$$\frac{\partial\Psi_2}{\partial\varphi} = 0 \quad \left(\varphi = 0, \frac{\pi}{l}\right), \qquad \Psi_2 = 0 \quad (\tau < \tau_*). \tag{7.5}$$

The solution given by the above system of equations must satisfy the following condition at the edge of the wedge:

$$\mathbf{u} = c + O(r^{\varepsilon}) \qquad (r \to 0, \varepsilon > 0, c = c(\tau, z)). \tag{7.6}$$

This condition ensures the integrability of the stresses (for $r \to 0$) and the uniqueness of the above problem (it is assumed that the condition (7.6) is uniformly satisfied in τ, φ, z). Besides, while solving the problem, it is assumed that $1/2 \leqslant l < 1$, since, as we shall see later, the solution for $l \geqslant 1$ may be obtained from the symmetric part of the solution (with respect to bisector plane of the wedge) obtained for $1/2 \leqslant l < 1$.

Before solving this problem, it should be mentioned that the potentials Φ, Ψ_1, and Ψ_2, being the solutions of the systems of equations (7.3)-(7.5), are not independent but are related through the condition (7.6). In view of this, certain difficulties arise in the process of obtaining solutions for this problem, as well as other dynamic problems for a wedge under mixed boundary conditions.

To begin with, we shall seek the solutions of the systems (7.3)-(7.5) independently as long as the condition on the wedge is not taken into account. For this purpose, we successively apply the Laplace double transformations in τ and z to these systems:

$$\Delta_1\bar{\Phi}^* = \omega^2\bar{\Phi}^* \qquad \left(\Delta_1 \equiv \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + r^{-2}\frac{\partial^2}{\partial\varphi^2}\right),$$
$$\frac{\partial\bar{\Phi}^*}{\partial\varphi} = -\frac{\partial\bar{\Phi}_0^*}{\partial\varphi} \qquad \left(\varphi = 0, \frac{\pi}{l}\right); \tag{7.7}$$

$$\Delta_1 \bar{\Psi}_1^* = \varkappa^2 \bar{\Psi}_1^*,$$
$$\bar{\Psi}_1^* = 0 \qquad \left(\varphi = 0, \frac{\pi}{l}\right); \tag{7.8}$$

$$\Delta_1 \bar{\Psi}_2^* = \varkappa^2 \bar{\Psi}_2^*,$$
$$\frac{\partial \bar{\Psi}_2^*}{\partial \varphi} = 0 \qquad \left(\varphi = 0, \frac{\pi}{l}\right); \tag{7.9}$$

$$\bar{\varphi}(p, r, \varphi, z) = \int_{-\infty}^{\infty} \varphi(\tau, r, \varphi, z) e^{-p\tau} d\tau$$
$$(\varphi = \Phi, \Psi_1, \Psi_2),$$
$$\bar{\varphi}^*(p, r, \varphi, s) = \int_{-\infty}^{\infty} \bar{\varphi}(p, r, \varphi, z) e^{-sz} dz$$
$$(\mathrm{Re}\, p > 0, |\mathrm{Re}\, s| < \mathrm{Re}\, p),$$
$$\bar{\Phi}_0^*(p, r, \varphi, s) = \bar{f}(p) e^{pr_0} \int_{-\infty}^{\infty} e^{-sz - pR} \frac{dz}{R} = \overline{2f(p)} e^{pr_0} K_0(p\omega),$$
$$f(\tau) \div \overline{f(p)}, \qquad \rho = [r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)]^{1/2},$$
$$\omega = \sqrt{p^2 - s^2}, \qquad \varkappa = \sqrt{\gamma^2 p^2 - \rho^2}.$$

Here, $K_\alpha(s)$ is the αth-order MacDonald function of the argument s (the inequality $\mathrm{Re}\, p > 0$ follows from the fact that in the Laplace double transformation, the integration with respect to τ is actually carried out for $\tau > -r_0$, since the wave source does not exist for $\tau < -r_0$). In order to isolate the single-valued branches of the functions ω and $\varkappa$, cuts are made in the plane s from the points $s = \pm p$ ($s = \pm \gamma p$ for $\varkappa$) to infinity along the rays $\arg s = \arg p$ and $\arg s = \arg p + \pi$. Besides, the branches ω and $\varkappa$ are chosen in such a way that $\omega = p$ and $\varkappa = \gamma p$ for $s = 0$. It can then be easily verified that $\mathrm{Re}\, \omega > 0$ and $\mathrm{Re}\, \varkappa > 0$ for $|\mathrm{Re}\, s| < \mathrm{Re}\, p$.

Next, in order to solve the systems (7.7)-(7.9), we expand the functions $\bar{\Phi}^*$ and $\bar{\Psi}_2^*$ into cosine series and the function $\bar{\Psi}_1^*$ into sine series in the interval $0 \leqslant \varphi \leqslant \pi/l$. To get the equations for determining the coefficients of these series, we multiply the equations for $\bar{\Phi}^*$ and $\bar{\Psi}_2^*$ from (7.7) and (7.9) by $2l\pi^{-1} \cos nl\varphi\, d\varphi$, and the equation for $\bar{\Psi}_1^*$ from (7.8) by $2l\pi^{-1} \sin nl\varphi\, d\varphi$, and integrate with respect to φ from 0 to π/l. As a result, taking into account the boundary conditions in the systems (7.7)-(7.9), we get the following second-order ordinary differential equations:

$$\Delta_2 a_n = \omega^2 a_n + f_n(r) \qquad \left(\Delta_2 \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2 l^2}{r^2}\right), \tag{7.10}$$

$$\Delta_2 b_{nj} = x^2 b_{nj} \qquad (j = 1, 2), \tag{7.11}$$

where we use the notation

$$f_n(r) = \frac{2l}{\pi r^2}\left[(-1)^n \frac{\partial\bar{\Phi}_0^*}{\partial\varphi}\bigg|_{\varphi=\frac{\pi}{l}} - \frac{\partial\bar{\Phi}_0^*}{\partial\varphi}\bigg|_{\varphi=0}\right],$$

$$\bar{\Phi}^*(p, r, \varphi, s) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nl\varphi,$$

$$a_n = \frac{2l}{\pi}\int_0^{\pi/l} \bar{\Phi}^* \cos nl\varphi\, d\varphi \qquad (n = 0, 1, 2, \ldots);$$

$$\bar{\Psi}_1^*(p, r, \varphi, s) = \sum_{n=1}^{\infty} b_{n1} \sin nl\varphi,$$

$$b_{n1} = \frac{2l}{\pi}\int_0^{\pi/l} \bar{\Psi}_1^* \sin nl\varphi\, d\varphi \qquad (n = 1, 2, 3, \ldots);$$

$$\bar{\Psi}_2^*(p, r, \varphi, s) = \frac{b_{02}}{2} + \sum_{n=1}^{\infty} b_{n2} \cos nl\varphi,$$

$$b_{n2} = \frac{2l}{\pi}\int_0^{\pi/l} \bar{\Psi}_2^* \cos nl\varphi\, d\varphi \qquad (n = 0, 1, 2, \ldots).$$

Solving (7.10) and (7.11), we get

$$a_n = A_n K_{nl}(r\omega) + B_n I_{nl}(r\omega) + F_n(r), \tag{7.12}$$

$$F_n(r) = -K_{nl}(r\omega)\int_0^r I_{nl}(x\omega) f_n(x) x\,dx - I_{nl}(r\omega)\int_r^{\infty} K_{nl}(x\omega) f_n(x) x\,dx,$$

$$b_{nj} = C_{nj} K_{nl}(rx) + D_{nj} I_{nl}(rx) \qquad (j = 1, 2). \tag{7.13}$$

Here, $I_\alpha(s)$ is the αth-order modified Bessel function of the first kind. Using the asymptotic expressions for the cylindrical functions

$$K_\alpha(s) \sim \left[\frac{\pi}{2s}\right]^{1/2} e^{-s}, \qquad I_\alpha(s) \sim (2\pi s)^{-1/2} e^s$$

for $|s| \to \infty$, $|\arg s| < \pi/2$, we find that $F_n(r) \to 0$ for $r \to \infty$. Since the perturbations propagate with a finite velocity, we assume that $|\bar{\Phi}| < cR^{-1}|e^{-pR}|$, $|\bar{\Psi}_j| < cR^{-1}|e^{-pR}|$ for $R \to \infty$, where c is independent of r, φ, and z. Hence,

$a_n \to 0$, $b_{nj} \to 0$ for $r \to \infty$. Then, we find from (7.12) and (7.13) that $B_n \equiv D_{nj} \equiv 0$. In order to determine the remaining coefficients A_n and C_{nj}, we use the condition (7.6). For this purpose, we apply the Laplace double transformation in τ and z. After this, we expand the transforms $\bar{u}_r^*$ and $\bar{u}_z^*$ into cosine series and u_φ^* into sine series in the interval $0 \leqslant \varphi \leqslant \pi/l$, using the expressions for the components of the displacement vector in terms of potentials as given by formulas (7.2). To get these expansions, we multiply the expressions for the components $\bar{u}_r^*$ and $\bar{u}_z^*$ by $2l\pi^{-1}\cos nl\varphi\, d\varphi$, the expressions for $\bar{u}_\varphi^*$ by $2l\pi^{-1}\sin nl\varphi\, d\varphi$, and integrate them with respect to φ between 0 and π/l. As a result, we get the following system of equations for (7.6) for each $n (n = 0, 1, 2, \ldots)$:

$$\begin{gathered}\frac{da_n}{dr} + \frac{nl}{r} b_{n1} + s\frac{db_{n2}}{dr} = c + O(r^\varepsilon), \\ sa_n - x^2 b_{n2} = c + O(r^\varepsilon)(\varepsilon > 0, r \to 0), \\ -\frac{nla_n}{r} - \frac{db_{n1}}{dr} - \frac{snl}{r} b_{n2} = c + O(r^\varepsilon).\end{gathered} \tag{7.14}$$

The coefficients A_n and C_{nj} appearing in the expressions (7.12) and (7.13) for a_n and b_{nj} can be determined from Eq. (7.14) (for $n = 0$, the system of three equations degenerates into a system of two equations in a_0 and b_{02}, since $b_{01} = 0$). In order to determine A_n and C_{nj}, we use the asymptotic expressions for the cylindrical functions $I_\alpha(s)$ and $K_\alpha(s)$ for $s \to 0$:

$$\begin{gathered}I_\alpha(s) = (s/2)^\alpha/\Gamma(1+\alpha) + O(s^{2+\alpha}), \\ K_0(s) = -\ln s + O(1), K_1(s) = s^{-1} + O(s\ln s), \\ 2K_\alpha(s) = \Gamma(\alpha)(2/s)^\alpha + \begin{cases}\Gamma(-\alpha)(s/2)^\alpha + O(s^{2-\alpha}) & (0 < \alpha < 1), \\ O(s^{2-\alpha}) & (\alpha > 1).\end{cases}\end{gathered} \tag{7.15}$$

With the help of these relations, it can be shown that the following asymptotic estimates are valid for $F_n(r)$ as $r \to 0$:

$$F_0(r) = c + O(r), \qquad F_1(r) = Mr^l + O(r), \tag{7.16}$$

where

$$M = -\left[\left(\frac{\omega}{2}\right)^l/\Gamma(1+l)\right]\int_0^\infty K_l(x\omega) f_1(x) x\, dx$$

$$= 8f(p)\left[e^{pr_0}/\Gamma(l)\right]K(r_0\omega)\left(\frac{\omega}{2}\right)^l \cos l\varphi_0,$$

$$F_n(r) = O(r) \quad (n \geqslant 2)$$

(in particular, $F_{2n}(r) \equiv 0$ for $l = 1/2$, since in this case $f_{2n}(r) \equiv 0$).

Substituting (7.12) and (7.13) into (7.14) and using the asymptotic estimates (7.15) and (7.16), we find that the conditions (7.14) will be satisfied for $n = 0$ and $n \geqslant 2$, if we put $A_n \equiv C_{nj} \equiv 0$. For $n = 1$, we get the following system from (7.14):

$$\begin{aligned} &Sr^{-l-1} + Tr^{l-1} + O(1) = c + O(r^\varepsilon),\\ &Wr^{-l} + O(r^l) = c + O(r^\varepsilon) \quad (\varepsilon > 0, r \to 0),\\ &Sr^{-l-1} - Tr^{l-1} + O(1) = c + O(r^\varepsilon), \end{aligned}$$

where we have used the notation

$$\begin{aligned} &S = -2^{l-1}\Gamma(1+l)[A_1\omega^{-l} - C_{11}\varkappa^{-l} + sC_{12}\varkappa^{-l}],\\ &T = Ml - 2^{-l-1}\Gamma(1-l)[A_1\omega^l + C_{11}\varkappa^l + s\varkappa^l C_{12}],\\ &W = 2^{l-1}\Gamma(l)[A_1 s\omega^{-l} - \varkappa^{2-l}C_{12}]. \end{aligned} \tag{7.17}$$

From (7.17) we find that $s = 0$, $T = 0$, and $W = 0$, which gives the following expressions for A_1, C_{11}, and C_{12}:

$$\begin{aligned} &A_1 = \omega^l\varkappa^{2-l}s^{-1}C_{12}, \qquad C_{11} = \gamma^2p^2s^{-1}C_{12},\\ &C_{12} = \frac{16}{\pi}\frac{l\bar{f}(p)s\omega^l K_l(r_0\omega)\sin l\pi\cos l\varphi_0}{\omega^{2l}\varkappa^{2-l} + (s^2+\gamma^2p^2)\varkappa^l}e^{pr_0}. \end{aligned} \tag{7.18}$$

As a result, using (7.18), we can get the following expressions for $\bar{\Phi}^*$, $\bar{\Psi}_1^*$ and $\bar{\Psi}_2^*$:

$$\begin{aligned} &\bar{\Phi}^* = \frac{1}{2}F_0(r) + \sum_{n=1}^{\infty} F_n(r)\cos nl\varphi + s^{-1}\varkappa^{2-l}\omega^l\cos l\varphi K_l(r\omega)C_{12},\\ &\bar{\Psi}_1^* = s^{-1}p^2\gamma^2\sin l\varphi K_l(r\varkappa)C_{12},\\ &\bar{\Psi}_2^* = \cos l\varphi K_l(r\varkappa)C_{12}. \end{aligned} \tag{7.19}$$

The sum of terms containing F_n $(n = 0, 1, 2, \ldots)$, which appears in the expression for $\bar{\Phi}^*$, is the Laplace transform (in τ and z) of the perturbed solution Φ' of the corresponding acoustic problem ($\mu = 0$). This can be proved quite easily if we note that, firstly, the original function of this sum satisfies the system of equations (7.7) and, secondly, it can be shown that this original function (for sufficiently smooth $f(\tau)$) satisfies the condition which ensures the uniqueness of the solution of the acoustic problem [53]:

$$\frac{\partial(\Phi_0 + \Phi')}{\partial\tau} = O(1), \qquad \frac{r\partial(\Phi_0 + \Phi')}{\partial r} = o(1) \qquad (r \to 0).$$

Thus, adding the transform of the incident spherical wave and applying the reverse Laplace transformation in s and p, we get ($\Phi_1 = \Phi_0 + \Phi$)

$$\Phi_l = \Phi_a + \frac{\cos l\varphi}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} e^{p\tau} dp \int\limits_{b_0 - i\infty}^{b_0 + i\infty} s^{-1} x^{2-c} \omega^l C_{12} K_l(r\omega) e^{sz} ds,$$

$$\Psi_l = \gamma^2 \frac{\sin l\varphi}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} e^{p\tau} p^2 dp \int\limits_{b_0 - i\infty}^{b_0 + i\infty} s^{-1} C_{12} K_l(rx) e^{sz} ds, \tag{7.20}$$

$$\Psi_2 = \frac{\cos l\varphi}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} e^{p\tau} dp \int\limits_{b_0 - i\infty}^{b_0 + i\infty} C_{12} K_l(rx) e^{sz} ds \quad (c_0 > |b_0|).$$

Taking into account the results given in [54], the acoustic solution $\Phi_a = \Phi_0 + \Phi'$ in this case may be expressed in the form

$$\Phi_a = \frac{f(+0)}{1 + \tau/r_0} Q\left(\tau + \frac{\tau^2 - r^2 - z^2}{2r_0}, r, \varphi\right)$$
$$- \int\limits_{R - r_0}^{\tau} \frac{d}{dx}\left[\frac{f(\tau - x)}{1 + x/r_0}\right] Q\left(x + \frac{x^2 - r^2 - z^2}{2r_0}, r, \varphi\right) dx, \tag{7.21}$$

where $Q(\tau, r, \varphi)$ is the solution of the acoustic problem on the diffraction of the plane wave $H[\tau + r\cos(\varphi - \varphi_0)] r_0^{-1}$ at the wedge under consideration. It has been shown in [53] that this solution can be presented in the following form:

$$Q(\tau, r, \varphi) = H(r - \tau)\{\sigma(\varphi - \varphi_0) H[\tau + r\cos(\varphi - \varphi_0)]$$
$$+ \sigma(\varphi - \varphi_0) H[\tau + r\cos(\varphi + \varphi_0)^*]\} r_0^{-1}$$
$$+ H(\tau - r)\pi^{-1} r_0^{-1}(\arctan\lambda_+ + \arctan\lambda_-),$$

$$\lambda_\pm = \frac{(1 - y^{2l})\sin l\pi}{(1 + y^{2l})\cos l\pi - 2y^l \cos l(\varphi \pm \varphi_0)},$$

$$y = \frac{\tau}{r} - \left[\left(\frac{\tau}{r}\right)^2 - 1\right]^{1/2},$$

$$\sigma(\varphi) = 1 \quad (|\varphi| < \pi), \quad \sigma(\varphi) = 0 \quad (\pi < |\varphi| < \pi/l),$$

$$\sigma(\varphi + 2\pi/l) = \sigma(\varphi), (\varphi + \varphi_0)^* = \varphi + \varphi_0 + 2\pi m/l,$$
$$H(x) = 1 \quad (x > 0), \quad H(x) = 0 \quad (x < 0).$$

The integer $m (m = 0, -1)$ is chosen in such a way that the equality $-\pi/l < (\varphi + \varphi_0)^* \leqslant \pi/l$ is always satisfied at a point in the physical space which we are considering.

It can be seen from (7.20) that the elastic terms supplementing the acoustic solution disappear only for $\varphi_0 = \pi/2l$ (the case of symmetry with respect to the bisector

plane of the wedge) and for $l \to 1$ (reflection of the wave at a plane wall). It can be verified with the help of (7.20) that in the vicinity of the edge of the wedge, the displacements are bounded and the stresses are integrable.

Using the change of variable $q = s/p$ and putting $b_0 = 0$, the expressions (7.20) can be represented as follows:

$$\Phi_1 = \Phi_* + \frac{\cos l\varphi}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} \overline{f(p)} e^{p(r_0 + \tau)} p\, dp \int\limits_L \xi(p, q)\, dq,$$

$$\Psi_1 = \frac{\gamma^2 \sin l\varphi}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} \overline{f(p)} e^{p(r_0 + \tau)} p\, dp \int\limits_L \zeta(p, q)\, dq,$$

$$\Psi_2 = \frac{\cos l\varphi}{(2\pi i)^2} \int\limits_{c_0 - i\infty}^{c_0 + i\infty} \overline{f(p)} e^{p(r_0 + \tau)} dp \int\limits_L \zeta(p, q) q\, dq, \tag{7.22}$$

$$\xi(p, q) = K_l(pr\sqrt{1 - q^2})(1 - q^2)^{\frac{l}{2}} M_0 e^{pqz},$$

$$\zeta(p, q) = K_l(pr\sqrt{\gamma^2 - q^2})(\gamma^2 - q^2)^{\frac{l}{2} - 1} M_0 e^{pqz},$$

$$M_0 \equiv M_0(p, q) = \frac{16}{\pi} l K_l(pr_0\sqrt{1 - q^2}) \frac{(1 - q^2)^{\frac{l}{2}} \sin l\pi \cos l\varphi_0}{(1 - q^2)^l + (q^2 + \gamma^2)(\gamma^2 - q^2)^{l - 1}}.$$

The contour L is shown in Fig. 59, where $\alpha_0 = \pi/2 - \arg p$, and the cuts in the plane s become the cuts along the real axis from the points $\pm\gamma$ and ± 1 to infinity in the plane q (and $(\gamma^2 - q^2)^{1/2} = \gamma$, $(1 - q^2)^{1/2} = 1$ for $q = 0$).

It is sufficient to consider the formulas (7.22) for $z \geqslant 0$, since Φ_1 and Ψ_1 are even functions of z, while Ψ_2 are odd (this can be easily proved with the help of (7.22)). Then, it can be shown for $z > 0$ that in the expression for Φ_1, the contour L may be deformed into the curve L_0, while in the expressions for Ψ_1 and Ψ_2, this contour can be deformed into L_1. Points on these curves satisfy the equations

$$\operatorname{Im}[qz - (r + r_0)(1 - q^2)^{1/2}] = 0 \qquad (\text{for } L_0),$$

$$\operatorname{Im}[qz - r(\gamma^2 - q^2)^{1/2} - r_0(1 - q^2)^{1/2}] = 0 \qquad (\text{for } L_1).$$

Both these curves, whose shape in the q-plane in shown in Fig. 59, are symmetric with respect to the real axis and have the same asymptotes away from the origin of coordinates. These asymptotes form angles $\pm\alpha_1$ with the real axis: $\tan\alpha_1 = -(r + r_0)/z$. The points q_0 and q_1 of intersection of L_0 and L_1 with the real axis are determined from the following equations respectively:

$$z + (r + r_0)q(1 - q^2)^{-1/2} = 0,$$

$$z + rq(\gamma^2 - q^2)^{-1/2} + r_0 q(1 - q^2)^{-1/2} = 0.$$

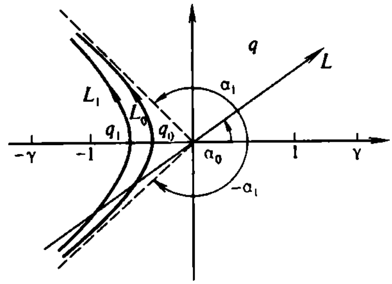

Fig. 59. The L_1 and L_0 contours.

Here, the function $N_0(q, z, r) = qz - (r + r_0)(1 - q^2)^{1/2}$, which is real on the curve L_0, assumes its maximum value on L_0 at the point q_0:

$$N(q_0, z, r) = -[z^2 + (r + r_0)^2]^{1/2}.$$

Similarly, the function

$$N_1(q, z, r) \equiv qz - r(\gamma^2 - q^2)^{1/2} - r_0(1 - q^2)^{1/2},$$

which is real on L_1, assumes its maximum value on L_1 at the point q_1:

$$N_1(q, z, r) = -R_1 \qquad (R_1 \equiv R_1(z, r)).$$

As $z \to 0$, the values of q and q_1 tend to zero, and the curves L_0 and L_1 in the limit turn into an imaginary axis. Consequently, the formulas (7.22) can be expressed for $z \geqslant 0$ in the following form:

$$\Phi_1 = \Phi_a + \cos l\theta \int_L M_1(1 - q^2)^{\frac{l}{2} - \frac{1}{4}} dq \int_{-0}^{x_0} f'(x) U(x, q) dx,$$

$$\Psi_1 = \gamma^2 \sin l\theta \int_{L_1} M_1(\gamma^2 - q^2)^{\frac{l}{2} - \frac{5}{4}} dq \int_{-0}^{x_1} f'(x) V(x, q) dx,$$

$$\Psi_2 = \cos l\theta \int_{L_1} M_1(\gamma^2 - q^2)^{\frac{l}{2} - \frac{5}{4}} q\, dq \int_0^{x_1} f(x) V(x, q) dx, \tag{7.23}$$

$$U(x, q) = P_{l - 1/2}\left[1 + \frac{(x_0 - x)^2 + 2(x_0 - x)(r + r_0)\sqrt{1 - q^2}}{2rr_0(1 - q^2)}\right],$$

$$V(x, q) = P_{l - 1/2}\left[1 + \frac{(x_1 - x)^2 + 2(x_1 - x)(r\sqrt{\gamma^2 - q^2} + r_0\sqrt{1 - q^2})}{2rr_0\sqrt{(1 - q^2)(\gamma^2 - q^2)}}\right],$$

$$M_1 = \frac{4l \sin l\pi \cos l\theta_0 (1 - q^2)^{\frac{l}{2} - \frac{1}{4}}}{\pi i \sqrt{rr_0}[(1 - q^2)^l + (q^2 + \gamma^2)(\gamma^2 - q^2)^{l - 1}]},$$

$$x_0 = \tau + r_0 + qz - (r + r_0)\sqrt{1 - q^2},$$

$$x_1 = \tau + r_0 + qz - r\sqrt{\gamma^2 - q^2} - r_0\sqrt{1 - q^2}.$$

Here, $P_{l - 1/2}(x)$ is a Legendre function of the first kind, and the derivative $f'(x)$ in the expressions for Φ_1 and Ψ_1 is taken in the generalized sense. The following inversion formula has been used while deriving Eqs. (7.23):

$$K_l(ps)K_l(pq)e^{p(s+q)} \doteqdot \frac{\pi}{2\sqrt{sq}} P_{l - 1/2}\left[\frac{(\tau + 2s)(\tau + 2q)}{2sq} - 1\right]H(\tau)$$

$$(|\arg s| < \pi, \ |\arg q| < \pi).$$

It should be noted that while deriving the relations (7.23), we have transformed the integration path in the q-plane to the contours L_0 and L_1 along which the functions $N_0(q, z, r)$ and $N_1(q_1, z, r)$ respectively are real. Hence, we have used the ideas behind Cagniard's method, which has been described in Sec. 4. In spite of the cumbersome nature of the expressions (7.23), they can be used to carry out a qualitative analysis of the results obtained. From (7.23), we find that

$$\Phi_1 = \Phi_a \quad \text{if} \quad \max x_0(q) < 0 \quad (\max x_0(q) = x_0(q_0) = \tau + r_0 - [z^2 + (r + r_0)^2]^{1/2}),$$

$$\Psi_1 = \Psi_2 = 0 \quad \text{if} \quad \max x_1(q) < 0 \quad (\max x_1(q) = x_1(q_1) = \tau + r_0 - R_1).$$

Consequently,

$$\tau + r_0 = [z^2 + (r + r_0)^2]^{1/2}, \quad \tau + r_0 = R_1$$

represent the equations of the wavefronts of diffracted longitudinal and transverse waves respectively. The patterns of the perturbed regions in the cross section $z = \text{const}$ $(\tau > (z^2 + r_0^2)^{1/2} - r_0)$ with and without shadows are shown in Figs. 60*a* and 60*b* respectively. Here, $\angle BOA = \pi - \theta_0$, $\angle COA = 2\pi/l - \pi - \theta_0$, $\angle GOA = \pi + \theta_0$ (the angles are measured from the ray OA in the counterclockwise direction), and the wavefronts *1-5* are given by the expressions (1) $\tau + r_0 = R(\theta)$, (2) $\tau + r_0 = R(-\theta)$, (3) $\tau + r_0 = \sqrt{z^2 + (r + r_0)^2}$ and (4) $\tau + r_0 = R_1$, (5) $\tau + r_0 = R(2\pi/l - \theta)$, $R(\theta) \equiv R = [z^2 + r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)]^{1/2}$ (the coordinate θ is measured from the ray OA in the counterclockwise direction, $\beta = 2\pi - \pi/l$). In the plane $z = 0$, the wavefronts of the diffracted longitudinal and transverse waves are given by the equations $r = \tau$ and $r = \tau/\gamma$ respectively.

It should be noted that the conditions at the edge lead primarily to a qualitative difference in the solution of the elastic problem from the solution of the corresponding acoustic problem ($\mu = 0$), since in addition to a supplementary diffracted longitudinal wave $\Phi_1 - \Phi_a$, we now have diffracted transverse waves of both types Ψ_1 and Ψ_2, differing in the direction of polarization of the displacement

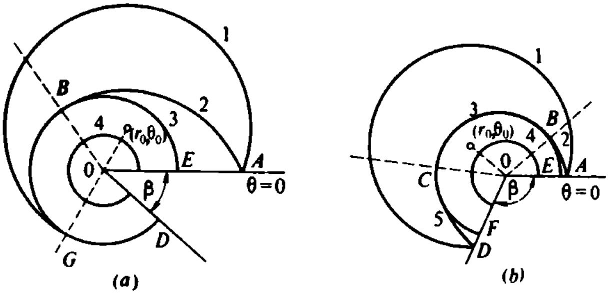

Fig. 60. The perturbed domains (*a*) with shadow, (*b*) without shadow.

vector. Additional perturbations $\Phi_1 - \Phi_a$, Ψ_1, and Ψ_2 describe the effect of elasticity.

If we put $f(\tau) = r_0 H(\tau)$ in (7.22) and if r_0 tends to infinity, we can show that $\Psi_2 \to 0$, and Φ_1 and Ψ_1 give, in the limit, the solution of the problem [55] of diffraction of a plane longitudinal step-shaped wave at a wedge:

$$\Phi_1 = \Phi_a^0 + \frac{4\sin l\pi \cos l\theta_0 \cos l\theta}{\pi(1+\gamma^{2l})}\left[P\left(\frac{\tau}{r}\right) - 1/P\left(\frac{\tau}{r}\right)\right]H(\tau - r),$$

$$\Psi_1 = \frac{4\gamma^2 \sin l\pi}{\pi(1+\gamma^{2l})}\cos l\theta_0 \sin l\theta\left[P\left(\frac{\tau}{r\gamma}\right) - 1/P\left(\frac{\tau}{r\gamma}\right)\right]H(\tau - r\gamma),$$

where $P(x) = [x + (x^2-1)^{1/2}]^l$, and Φ_a^0 is the solution of the corresponding acoustic problem.

Let us consider in greater detail the most interesting case of an incident wave (7.1), when $f(\tau) = -\dfrac{r_0}{2}\tau^2 H(\tau)$. The stress at the front of such a wave undergoes a finite jump:

$$[\sigma_n] = \sigma_n^+ - \sigma_n^- = -(\lambda + 2\mu)r_0(\tau + r_0)^{-1},$$

where n is the normal to the wavefront, and the plus and minus signs correspond to the regions behind and in front of the wavefront respectively; as r_0 tends to infinity, this wave turns into a plane wave whose potential is given by

$$\Phi_0 = -\frac{1}{2}[\tau + r\cos(\theta_0 - \theta)]^2 H[\tau + r\cos(\theta - \theta_0)].$$

In order to investigate the solution, it is sufficient to consider the case $0 < \theta_0 < \pi/l - \pi$ (see Fig. 60*a*), which covers all the possible regions of perturbed motion: the region of reflected wave ($0 < \theta < \pi - \theta_0$, $[(\tau + r_0)^2 - z^2]^{1/2} - r_0 < r$, $\tau + r_0 > R(-\theta)$), the region of diffraction ($r < [(\tau +$

$+ r_0)^2 - z^2]^{1/2} - r_0)$, and the region of shadow ($\pi + \theta_0 < \theta < \pi/l$). While going over through the front *2* of the reflected wave, the incident wave potential is supplemented by the potential of the reflected wave

$$\Phi = -r_0[\tau + r_0 - R(-\theta)]^2 H[\tau + r_0 - R(-\theta)]\,\frac{1}{2R(-\theta)}$$

and on the reflected wavefront there appears a finite jump of normal pressure equal to $[\sigma_n] = -r_0(\lambda + 2\mu)(\tau + r_0)^{-1}$ (since $\partial^2\Phi/\partial n^2$ undergoes discontinuity). While going over through the front *3* of the longitudinal diffracted wave, strains, and consequently stresses, are continuous and the derivative $\partial\varepsilon_n/\partial n$ of the strain with respect to the normal to the wavefront undergoes a second-order discontinuity (since $\partial^3\Phi_1/\partial n^3$ undergoes such a discontinuity). Upon approaching the wavefront from the region in front of it, this derivative has a singularity of the order of $\varepsilon^{-1/2}$. The acoustic solution Φ_a and the additional elastic term in the solution also have singularities of the order $\varepsilon^{-1/2}$. On the transverse wavefront *4*, the strains are continuous, while the derivatives $\partial\varepsilon_{\theta n}/\partial n$ and $\partial\varepsilon_{\nu n}/\partial n$ of the strain components with respect to the normal undergo a second-order discontinuity (here, ν is measured along the line of intersection of the transverse wavefront with the plane θ = const), since the derivatives $\partial^3\Psi_1/\partial n^3$ and $\partial^4\Psi_2/\partial n^4$ undergo a discontinuity at this front. Moreover, these derivatives are finite if we approach the front from outside ($\tau + r_0 < R_1$), and have a singularity of the order of $\varepsilon^{-1/2}$ if we approach the front from inside ($\tau + r_0 > R_1$).

It can be seen from the above discussion that the additional elastic part of the solution is comparable in magnitude to the diffractional part of the acoustic solution not only in the neighbourhood of the edge of the wedge, but also in the vicinity of the front of the diffracted wave *3*. Consequently, the elastic problem is considerably different from the acoustic problem not only in the vicinity of the edge, but, generally speaking, over the entire domain of diffraction $r + r_0 < [(\tau + r_0)^2 - z^2]^{1/2}$.

From the solution obtained for angles $\beta = 2\pi - \pi/l$ ($1/2 \leqslant l < 1$) of the wedge satisfying the condition $\beta < \pi$, we can obtain solutions for angles $\beta \geqslant \pi$. Indeed, the part of the solution (7.22), which is symmetric with respect to the bisector plane of the wedge (denoted by Φ_1^s and Φ_j^s, $j = 1, 2$), satisfies the conditions $\dfrac{\partial\Phi_1^s}{\partial\theta} = \Psi_1^s = \dfrac{\partial\Psi_r^s}{\partial\theta} = 0$ on the bisector plane, and hence gives the solution of the problem of the diffraction of an elastic spherical wave at a wedge of angle $\beta_1 = 2\pi - \pi/l_1$, where $l_1 = 2l$ ($1 \leqslant l_1 < 2$). Here, the angle β_1 satisfies the conditions $\pi \leqslant \beta_1 < 3\pi/2$. Isolating once again the symmetric part of the solution for a wedge of angle β_1, we can obtain a solution for an angle $\beta_2 = 2\pi - \pi/l_2$ ($2 \leqslant l_2 = 2l_1 < 4$) which satisfies the inequality $3\pi/2 \leqslant \beta_2 < 7\pi/4$, and so on. After nth operation, we get the solution for an angle $\beta_n = 2\pi - \pi/l_n$ ($2^{n-1} \leqslant l_n = 2^n l < 2^n$) of the wedge, satisfying the inequality

$$2\pi - \frac{\pi}{2^{n-1}} \leqslant \beta_n < 2\pi - \frac{\pi}{2^n}\,.$$

Thus, we can get a solution for any angle β of the wedge in the interval $0 \leqslant \beta < 2\pi$. It follows from [52] that the additions to the acoustic solution vanish.

Section 8
Interaction of a Shear Wave with a Cylindrical Inclusion

Following [56], let us consider the problem of the action of a plane harmonic shear wave on a hard cylindrical inclusion linked with an elastic medium on a part of the surface.

Suppose that we have a cylindrical cavity of radius a in an elastic medium. The axis of this cylindrical cavity is taken as the z-axis in cylindrical and Cartesian systems of coordinates. A rigid cylinder is placed in the cavity. There is a complete linking in the region $r = a$, $|\theta| > \alpha$, $-\infty < z < \infty$, while the stresses vanish in the rest of the region $r = a$, $|\theta| < \alpha$.

Suppose that the solution at infinity has the form

$$w^* = w_0 e^{-i\omega t - ikx} \quad \left(k = \frac{\omega}{b}\right) \tag{8.1}$$

(the components u^* and v^* vanish). Then, for the complex amplitude at infinity, we have

$$\bar{w}^* = w_0 e^{-ikr\cos\theta}. \tag{8.1'}$$

In view of the conditions at infinity and the boundary conditions at the inclusion, we are justified in assuming that only the component w is non-zero everywhere in the elastic body. This component must satisfy the equation

$$\nabla^2 w - \frac{1}{b^2}\frac{\partial^2 w}{\partial t^2} = 0 \quad \left(\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right). \tag{8.2}$$

The corresponding equation for the amplitude has the form

$$\nabla^2 \bar{w} + k^2 \bar{w} = 0. \tag{8.2'}$$

We shall seek the solution of this problem in the form of the sum $w + w^*$, where the displacement w must satisfy the condition (8.2) as before, as well as the radiation conditions (1.15), Ch. 3, Vol. 1. The solution for the amplitude $\bar{w}$ may be taken in the form of the series

$$\bar{w} = A_0 \frac{H_0^{(1)}(kr)}{H_0^{(1)}(ka)} + \sum_{n=1}^{\infty} A_n \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \cos n\theta, \tag{8.3}$$

where $H_n^{(1)}(x)$ is a Hankel function.

We shall assume that the rigid inclusion remains stationary. This leads to the following condition at the contour $r = a$:

$$\bar{w}(a, \theta) = -\bar{w}^*(a, \theta) \qquad (\alpha < \theta \leqslant \pi). \tag{8.4}$$

Besides, the following condition must, of course, be satisfied:

$$\bar{\tau}_{rz}(a,\theta) = -\bar{\tau}^*_{rz}(a,\theta)\,(0 < \theta \leqslant \alpha). \tag{8.5}$$

In accordance with (8.3), we get the following representation for $\bar{\tau}_{rz}(r,\theta)$:

$$\frac{1}{\mu}\bar{\tau}_{rz}(r,\theta) = -A_0\frac{kH_1^{(1)}(kr)}{H_0^{(1)}(ka)} + \sum_{n=1}^{\infty} A_n\left[-\frac{kH_{n+1}^{(1)}(kr)}{H_n^{(1)}(ka)} + \frac{nH_n^{(1)}(kr)}{rH_n^{(1)}(ka)}\right]\cos n\theta. \tag{8.6}$$

The representations (8.3) and (8.6) enable us to rewrite the boundary conditions (8.4) and (8.5) in the form of a pair of series equations

$$-A_0\frac{kaH_1^{(1)}(ka)}{H_0^{(1)}(ka)} + \sum_{n=1}^{\infty} A_n\left[n - \frac{kaH_{n+1}^{(1)}(ka)}{H_1^{(1)}(ka)}\right]\cos n\theta = ikaw_0e^{-ika\cos\theta} \quad (0 \leqslant \theta \leqslant \alpha), \tag{8.7}$$

$$A_0 + \sum_{n=1}^{\infty} A_n\cos n\theta = -w_0e^{-ika\cos\theta} \quad (\alpha < \theta \leqslant \pi). \tag{8.8}$$

In order to solve these equations, we make use of the fact that an exact solution exists for one special class of pair of series equations (5.13), Ch. 1, Vol. 1.

We denote the left-hand side of Eq. (8.7) for $\alpha < \theta \leqslant \pi$ through

$$\frac{a}{\mu}\bar{\tau}(\theta) + ikaw_0\cos\theta e^{-ika\cos\theta}.$$

Here, $\bar{\tau}(\theta)$ is the amplitude of shearing stresses on the contour $r = a, \alpha < \theta \leqslant \pi$, corresponding to the field of the perturbed wave. Equation (8.7) can be replaced by an approximate equality of the type (N is a certain given number)

$$A_0\frac{kaH_1^{(1)}(ka)}{H_0^{(1)}(ka)} + \sum_{n=1}^{\infty} nA_n\cos n\theta + \sum_{n=1}^{N} A_n\left[\frac{kaH_{n+1}^{(1)}(ka)}{H_n^{(1)}(ka)} - 2n\right]\cos n\theta = \begin{cases} -ikaw_0\cos\theta e^{-ika\cos\theta} & (0 \leqslant \theta < \alpha), \\ -\dfrac{1}{\mu}\bar{\tau}(\theta) - ikaw_0\cos\theta e^{-ika\cos\theta} & (\alpha < \theta \leqslant \pi). \end{cases} \tag{8.9}$$

It is useful to note the equality

$$\lim_{k\to 0}\left[\frac{kaH_{n+1}^{(1)}(ka)}{H_n^{(1)}(ka)} - 2n\right] = 0. \tag{8.10}$$

Expanding the right-hand side of (8.9) into a Fourier series, we obtain the follow-

ing expressions for the coefficients:

$$A_0 \frac{kaH_1^{(1)}(ka)}{H_0^{(1)}(ka)} = -kaw_0J_1(ka) - \frac{a}{\pi\mu}\int_\alpha^\pi \bar{\tau}(\theta)d\theta,$$

$$A_n = -\frac{2a}{\pi\mu n}(1 - h_n)\int_\alpha^\pi \bar{\tau}(t)\cos nt\,dt$$

$$- w_0ka\frac{2}{n}(-1)^{n-1}i^nJ_n'(ka)(1 - h_n) \quad (n = 1, 2, \ldots, N), \quad (8.11)$$

$$A_n = -\frac{2a}{\pi\mu n}\int_\alpha^\pi \bar{\tau}(t)\cos nt\,dt - w_0ka\frac{2}{n}(-1)^{n-1}i^nJ_n'(ka) \qquad (n > N).$$

The following notation has been used here:

$$h_n = \frac{kaH_{n+1}^{(1)}(ka) - 2nH_n^{(1)}(ka)}{kaH_{n+1}^{(1)}(ka) - nH_n^{(1)}(ka)}.$$

While deriving the above formulas, we have used the identity

$$\int_0^\pi \cos t\cos nte^{-ika\cos t}dt = \pi(-1)^{n-1}i^{n-1}J_n'(ka).$$

Substituting (8.11) into Eq. (8.8) and interchanging the order of integration and summation, we get the equality

$$\frac{2a}{\pi\mu}\int_\alpha^\pi \bar{\tau}(t)\left(\sum_{n=1}^\infty \frac{\cos nt\cos n\theta}{n}\right)dt$$

$$-\frac{2a}{\pi\mu}\sum_{n=1}^N \frac{h_n}{n}\left(\int_\alpha^\pi \bar{\tau}(t)\cos nt\,dt\right)\cos n\theta$$

$$= A_0 + w_0J_0(ka) + 2w_0ka\sum_{n=1}^\infty (-i)^n\frac{J_{n-1}(ka)}{n}\cos n\theta$$

$$- 2w_0ka\sum_{n=1}^N (-i)^n\frac{h_n}{n}J_n'(ka)\cos n\theta. \quad (8.12)$$

While deriving this equation, we have used the identity

$$e^{-ika\cos\theta} = J_0(ka) + 2\sum_{n=1}^\infty (-i)^nJ_n(ka)\cos n\theta.$$

Following [57], we introduce new variables ξ and φ as per formulas

$$\cos t = \beta + \beta' \cos\xi, \cos\theta = \beta + \beta' \cos\varphi,$$
$$\beta = \frac{1}{2}(\cos\alpha - 1), \beta' = \frac{1}{2}(\cos\alpha + 1). \tag{8.13}$$

It is obvious that as the variables ξ and φ change in the interval $(0, \pi)$, the variables t and θ change in the interval (α, π). It follows from (8.13) that

$$\frac{dt}{d\xi} = \frac{\sqrt{\cos\alpha - \cos t}}{\sqrt{2}\sin\frac{t}{2}}, \frac{d\theta}{d\varphi} = \frac{\sqrt{\cos\alpha - \cos\theta}}{\sqrt{2}\sin\frac{\theta}{2}} \tag{8.14}$$
$$(\alpha < t < \pi, \alpha < \theta < \pi).$$

The function $\bar{\tau}(t(\xi))\dfrac{dt}{d\xi}$ may be represented in the form of a Fourier series in the interval $0 \leqslant \xi \leqslant \pi$:

$$\bar{\tau}(t)\frac{dt}{d\xi} = \mu \sum_{n=1}^{\infty} \alpha_n \cos(n-1)\xi. \tag{8.15}$$

Taking (8.14) into account, we get

$$\bar{\tau}(\theta)|_{\theta > \alpha} = \mu \frac{\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{\cos\alpha - \cos\theta}} \sum_{n=1}^{\infty} \alpha_n \cos(n-1)\varphi, \tag{8.16}$$

where

$$\varphi = \arccos\left(\frac{\cos\theta - \beta}{\beta_1}\right).$$

Using the expansion (8.15) and the equality

$$\sum_{n=1}^{\infty} \frac{\cos nt \cos n\theta}{n} = -\frac{1}{2}\ln|\cos t - \cos\theta|$$
$$= -\frac{1}{2}\ln\beta' + \sum_{n=1}^{\infty} \frac{\cos n\xi \cos n\varphi}{n},$$

the first component of the right-hand side of (8.12) may be represented in the form

$$\frac{2a}{\pi\mu}\int_{\alpha}^{\pi} \bar{\tau}(t)\left(\sum_{n=1}^{\infty} \frac{\cos nt \cos n\theta}{n}\right) dt$$

$$= \frac{2a}{\pi} \int_0^{\pi} \left(\sum_{n=1}^{\infty} \alpha_n \cos(n-1)\xi \right)$$

$$\times \left(-\frac{1}{2} \ln \beta' + \sum_{n=1}^{\infty} \frac{\cos n\varphi \cos n\xi}{n} \right) d\xi$$

$$= a \sum_{m=1}^{\infty} b_m \alpha_m \cos(m-1)\varphi, \quad (8.17)$$

where

$$b_1 = -\ln \beta', \quad b_m = \frac{1}{m-1} \quad (m = 2, 3, \ldots).$$

In order to transform the second component on the right-hand side of (8.12), we use the following expressions:

$$\begin{aligned} \cos nt &= \cos nt(\xi) = \sum_{m=1}^{n+1} c_m^{(n)} \cos(m-1)\xi, \\ \cos n\theta &= \cos n\theta(\varphi) = \sum_{m=1}^{n+1} c_m^{(n)} \cos(m-1)\varphi. \end{aligned} \quad (8.18)$$

Since

$$\cos n\theta = \cos n[\arccos(\beta + \beta' \cos\varphi)] = T_n(\beta + \beta' \cos\varphi),$$

where $T_n(x)$ is Chebyshev's polynomial, the coefficients $c_m^{(n)}$ can be easily determined with the help of the corresponding expressions for the polynomials $T_n(x)$. In particular, for $n = 1$ and $n = 2$, we get

$$c_1^{(1)} = \beta, c_2^{(1)} = \beta', c_1^{(2)} = (2\beta^2 - 1 + \beta'^2), c_2^{(2)} = 4\beta\beta', c_3^{(2)} = \beta'^2.$$

Thus, taking into account Eqs. (8.18) and (8.15), we get

$$\frac{2a}{\pi\mu} \int_{\alpha}^{\pi} \bar{\tau}(t) \cos nt \, dt = \frac{2a}{\pi\mu} \int_0^{\pi} \bar{\tau}(t(\xi)) \frac{dt}{d\xi} \cos nt(\xi) d\xi$$

$$= a \sum_{j=1}^{n+1} \varepsilon_j \alpha_j c_j^{(n)}, \quad \varepsilon_1 = 2, \varepsilon_j = 1 \quad (j = 2, 3, \ldots). \quad (8.19)$$

Substituting (8.17) and (8.19) into (8.12) and taking Eqs. (8.18) into consideration, we can rewrite Eq. (8.12) in the following form:

$$\sum_{m=1}^{\infty} b_m \alpha_m \cos(m-1)\varphi - \sum_{m=1}^{N+1} \left(\sum_{j=1}^{N+1} \alpha_j D_{jm} \right) \cos(m-1)\varphi$$

$$= \frac{1}{a}\,[A_0 + w_0 J_0(ka)] + \frac{2w_0}{a}\sum_{m=1}^{\infty} P_m \cos(m-1)\varphi$$

$$- \frac{2w_0}{a}\sum_{m=1}^{N+1} Q_m \cos(m-1)\varphi. \quad (8.20)$$

Here, we have used the notation

$$D_{jm} = \varepsilon_j \sum_{n=1}^{N} \frac{h_n}{n}\, c_j^{(n)} c_m^{(n)},$$

$$P_m = -\sum_{n=1}^{\infty} (-i)^n \frac{ka}{n} J_{n-1}(ka) c_m^{(n)},$$

$$Q_m = -\sum_{n=1}^{N} (-i)^n \frac{h_n}{n}\, ka J_n'(ka) c_m^{(n)}.$$

Since the variable φ changes between 0 and π, we get from Eq. (8.20) a system of $N + 1$ equations for determining the constants α_m:

$$b_m\alpha_m - \sum_{j=1}^{N+1} \alpha_j D_{jm} = \frac{2w_0}{a}(Q_m - P_m) + \left(\frac{w_0}{a} J_0(ka) + \frac{A_0}{a}\right)\delta_{1m}$$

$$(m = 1, 2, \ldots, N+1),\ \delta_{11} = 1,\ \delta_{1m} = 0 \quad (m = 2, 3, \ldots), \quad (8.21)$$

$$b_m\alpha_m = -2\frac{w_0}{a} P_m \qquad (m > N+1).$$

Substituting (8.15) into (8.11), we get

$$A_0 = -w_0 \frac{J_1(ka)H_0^{(1)}(ka)}{H_1^{(1)}(ka)} - a\,\frac{H_0^{(1)}(ka)}{kaH_1^{(1)}(ka)}\,\alpha_1. \quad (8.22)$$

Taking into account this equation, we can rewrite the system (8.21) in the form

$$b_m'\alpha_m - \sum_{j=1}^{N+1} \alpha_j D_{jm}$$

$$= 2\frac{w_0}{a}(Q_m - P_m) + \frac{w_0}{a}\left[J_0(ka) - \frac{J_1(ka)H_0^{(1)}(ka)}{H_1^{(1)}(ka)}\right]\delta_{1m}. \quad (8.23)$$

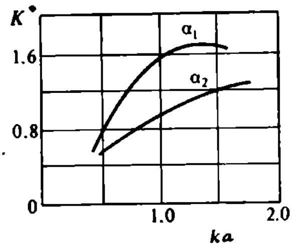

Fig. 61. Dependence of K^* on the wave number ka.

Here,

$$b_1' = -\ln\beta' + \frac{H_0^{(1)}(ka)}{kaH_1^{(1)}(ka)},$$

$$b_m' = b_m = \frac{1}{m-1} \quad (m = 2, 3, \ldots, N+1).$$

The solution of the pair of equations (8.7) and (8.8) obtained in this way is approximate and depends on the number N. Obviously, the approximate solution may become as close as possible to the exact solution with increasing N.

It is interesting to determine the dynamic stress intensity factor at the tip of a crack passing along the interface between a rigid inclusion and an elastic medium.

It can be easily seen that the part of the expression for the total stress field, which contains a singularity, can be obtained from the relation (8.16). Consequently, the dynamic stress intensity factor is given by

$$K_3 = K_3' + iK_3'' = \sqrt{2\pi a} \lim_{\theta \to \alpha} [\sqrt{\theta - \alpha}\,\bar{\tau}(\theta)] = \mu \sqrt{2\pi a \tan\frac{\alpha}{2}} \sum_{n=1}^{\infty} \alpha_n. \quad (8.24)$$

Figure 61 shows the calculated values of the quantity $K^* = \dfrac{|K_3|}{\mu\sqrt{2\pi a}} \dfrac{a}{w_0}$ as a function of the dimensionless wave number ka for $\alpha = \alpha_1 = \pi/4$ and $\alpha = \alpha_2 = \pi/18$. A special feature of this dependence is the existence of a maximum for K^* at $ka \approx 1.25$ ($\alpha = \pi/4$), which is typical of this class of problems [57, 58].

Section 9
Torsion and Stretching of a Cylinder with an External Circular Cut

1. Suppose that a solid circular cylinder of unit radius, having a circular cut of inner diameter $2a$, is twisted by the moments M (Fig. 62) [59]. The cylinder is con-

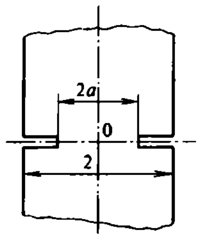

Fig. 62. A cylinder with a circular cut.

sidered in a cylindrical system of coordinates (r, θ, z) having origin in the plane of the cut. It is assumed that the lateral surface of the cylinder as well as the surface of the cut are free of stresses.

In the basic solution of the problem of twisting of a shaft of unit radius, the shearing stresses are given by the expressions

$$\tau_{\theta z}^{(0)} = \frac{2M}{\pi} r, \quad \tau_{r\theta}^{(0)} = 0. \tag{9.1}$$

In order that the surface of the cut be free of stresses, it is necessary to consider an additional state of stress which does not depend on the angular coordinate θ and is characterized by the only non-zero component of displacements, $u_\theta = u(r, z)$, satisfying the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{9.2}$$

The shearing stresses are determined according to the formulas

$$\tau_{\theta z}^{(1)} = \mu \frac{\partial u}{\partial z}, \quad \tau_{r\theta}^{(1)} = \mu \left(\frac{\partial u}{\partial r} - \frac{u}{r}\right). \tag{9.3}$$

We consider a semi-infinite cylinder $(z \geqslant 0)$ and write the solution of Eq. (9.2) in the form

$$u(r, z) = 2 \sum_{n=1}^{\infty} \frac{B_n J_1(\lambda_n z)}{\lambda_n} e^{-\lambda_n r}. \tag{9.4}$$

Here, $J_1(\lambda_n z)$ is the Bessel function of the first kind, and λ_n are the roots of the equation

$$\lambda_n J_1'(\lambda_n) - J_1(\lambda_n) = -\lambda_n J_2(\lambda_n) = 0.$$

The shearing stresses at the lateral surface are equal to zero, while at the end face, they are described in accordance with Eqs. (9.3) and (9.4) as follows:

$$\tau_{\theta z}^{(1)} = \mu \frac{\partial u}{\partial z} = -2\mu \sum_{n=1}^{\infty} B_n J_1(\lambda_n r), \quad z = 0. \tag{9.5}$$

In the plane $z = 0$, the boundary conditions are given by

$$\begin{gathered} \tau_{\theta z}^{(1)}(r, 0) = -\tau_{\theta z}^{(0)}(r, 0) \quad (a < r \leqslant 1), \\ u(r, 0) = 0 \quad (0 \leqslant r < a). \end{gathered} \tag{9.6}$$

Taking into account these boundary conditions and using the relations (9.4) and (9.5), we get the following pair of series equations:

$$\sum_{n=1}^{\infty} \frac{B_n}{\lambda_n} J_1(\lambda_n r) = 0 \quad (0 \leqslant r < a), \tag{9.7}$$

$$\sum_{n=1}^{\infty} B_n J_1(\lambda_n r) = \frac{M}{\pi\mu} r \quad (a < r \leqslant 1). \tag{9.8}$$

Following [60], we put

$$-\frac{M}{\pi\mu} r + \sum_{n=1}^{\infty} B_n J_1(\lambda_n r) = -\frac{\partial}{\partial r} \int_r^a \frac{g(t)dt}{\sqrt{t^2 - r^2}} \quad (r < a). \tag{9.9}$$

Then, on the basis of Eqs. (9.8) and (9.9), we get

$$B_n = 2J_1^{-2}(\lambda_n) \int_0^a g(u) \sin(\lambda_n u) du, \tag{9.10}$$

$$-\frac{M}{\pi\mu} = 8 \int_0^a u g(u) du. \tag{9.11}$$

The function $g(u)$ satisfies Fredholm's integral equation of the second kind with a symmetric kernel

$$g(t) = \int_0^a g(u) K(u, t) dt + \frac{16}{\pi} t \int_0^a u g(u) du, \tag{9.12}$$

$$K(u, t) = \frac{4}{\pi^2} \int_0^{\infty} \frac{K_2(y)}{I_2(y)} [8ut I_2(y) - \sinh(ty)\sinh(uy)] dy. \tag{9.13}$$

Here, $I_n(y)$ and $K_n(y)$ are the modified Bessel functions of the first and second kinds respectively.

Using the expansions into power series for the functions $\sinh(ty)$, $\sinh(uy)$, and $I_2(y)$, we can represent the kernel of Eq. (9.12) in the form

$$K(u,t) = -\sum_{k=0}^{\infty} t^{2k+1} b_{2k+1}(u), \tag{9.14}$$

$$b_1(u) = \sum_{n=2}^{\infty} \alpha_n \left[\frac{u^{2n-1}}{(2n-1)!} - \frac{u}{2^{2n-3}(n-1)!(n+1)!}\right],$$

$$b_{2k+1}(u) = \frac{1}{(2k+1)!} \sum_{n=2}^{\infty} \frac{\alpha_{n-k-1} u^{2n-3}}{(2n-3)!} \quad (k = 1, 2, \ldots), \tag{9.15}$$

$$\alpha_n = \frac{4}{\pi^2} \int_0^{\infty} \frac{K_2(y)}{I_2(y)} y^{2n} dy \quad (n = 2, 3, \ldots). \tag{9.16}$$

Taking into account the expansion (9.14), we shall seek the solution of the integral equation (9.12) in the form

$$g(t) = C(a) \sum_{m=0}^{\infty} P_{2m+1} t^{2m+1} \tag{9.17}$$

The constants P_{2m+1} in this equation are determined from the solution of the infinite system of algebraic equations

$$P_{2m+1} = -\sum_{k=0}^{\infty} P_{2k+1} C_{2k+1,2m+1} + \delta_m^0 \quad (m = 0, 1, 2, \ldots). \tag{9.18}$$

Here,

$$C_{2k+1,1} = \sum_{n=2}^{\infty} \alpha_n \left[\frac{a^{2k+2n+1}}{(2k+2n+1)(2n-1)!} - \frac{a^{2k+3}}{2^{2n-3}(2k+3)(n-1)!(n+1)!}\right],$$

$$C_{2k+1,2m+1} = \int_0^a u^{2k+1} b_{2m+1}(u) du = \frac{1}{(2m+1)!} \sum_{n=2}^{\infty} \frac{\alpha_{n+m-1} a^{2k+2n-1}}{(2k+2n+1)(2n-3)!} \tag{9.19}$$

$$(k = 0, 1, 2, \ldots; m = 1, 2, 3, \ldots), \quad \delta_m^0 = \begin{cases} 1, & m = 0, \\ 0, & m = 1, 2, \ldots. \end{cases}$$

The constant $C(a)$ is determined from (9.11) and the expansion (9.17):

$$C(a) \sum_{m=0}^{\infty} P_{2m+1} \frac{a^{2m+3}}{2m+3} = -\frac{M}{8\pi\mu}. \tag{9.20}$$

The system (9.18) is quasi-regular (see Sec. 15, Ch. 1, Vol. 1). In order to prove this, we shall first prove that

$$\lim_{m \to \infty} b_{2m+1}(u) = 0 \quad (0 \leqslant u < 1). \tag{9.21}$$

We substitute the expression for α_n into the series defining $b_{2m+1}(u)$, and change the order of summation and integration. This gives

$$b_{2m+1}(u) = \frac{4}{\pi^2(2m+1)!} \int_0^{\infty} \frac{K_2(y)}{I_2(y)} y^{2m+1} \sinh(uy)dy \qquad (m = 1, 2, \ldots).$$

Using the asymptotic expansions for the modified Bessel functions, we get for large values of m

$$b_{2m+1}(u) \sim \frac{2}{\pi(2-u)^{2m+1}} \left[1 + \frac{15(2-u)}{4(2m+1)} + \frac{225(2-u)^2}{32(2m+1)2m} + \cdots\right]$$
$$- \frac{2}{\pi(2+u)^{2m+1}} \left[1 + \frac{15(2+u)}{4(2m+1)} + \frac{225(2+u)^2}{32(2m+1)2m} + \cdots\right]. \tag{9.22}$$

This leads to Eq. (9.21).

It should be noted that for large values of n, the coefficients α_n may also be determined from the asymptotic formula

$$\alpha_n \sim \frac{(2n)!}{2^{2n-1}\pi} \left[1 + \frac{15}{4}\frac{2}{2n} + \frac{225}{32}\frac{2^2}{(2n-1)(2n)} + \cdots\right]. \tag{9.23}$$

Table 5 gives the values of α_n obtained in accordance with (9.16) (first row) and (9.23).

A comparison shows that for $n \geqslant 5$, the exact values of α_n differ only insignificantly from the asymptotic values.

TABLE 5

2	3	4	5	6	7	8
$117\cdot10^{-1}$	$297\cdot10^{-1}$	276	$498\cdot10^{1}$	$142\cdot10^{3}$	$587\cdot10^{4}$	$326\cdot10^{6}$
$80\cdot10^{-1}$	$274\cdot10^{-1}$	267	$489\cdot10^{1}$	$141\cdot10^{3}$	$584\cdot10^{4}$	$327\cdot10^{6}$
9	**10**	**11**	**12**	**13**	**14**	**15**
$235\cdot10^{8}$	$212\cdot10^{10}$	$235\cdot10^{12}$	$312\cdot10^{14}$	$490\cdot10^{16}$	$894\cdot10^{18}$	$189\cdot10^{21}$
$237\cdot10^{8}$	$216\cdot10^{10}$	$241\cdot10^{12}$	$323\cdot10^{14}$	$511\cdot10^{16}$	$946\cdot10^{18}$	$202\cdot10^{21}$

Evaluating the coefficients of the infinite system (9.18), we find

$$|C_{2k+1,2m+1}| \leqslant |b_{2m+1}(a_1)| \frac{a^{2k+2}}{2k+2} \quad (0 \leqslant a_1 \leqslant a < 1).$$

Consequently,

$$S_{2m+1} = \sum_{k=1}^{\infty} |C_{2k+1,2m+1}| \leqslant \frac{1}{2} |b_{2m+1}(a_1)| \, |\ln(1-a^2) + a^2|. \qquad (9.24)$$

It follows from the relations (9.23) and (9.24) that starting from a certain number $m = m'$, the following inequality holds:

$$S_{2m+1} < 1, \quad m \geqslant m'.$$

Thus, the system (9.18) is quasi-regular for $0 \leqslant a < 1$, and the constant m' is the larger, the closer a is to unity.

In order to determine the shearing stresses on the extension of a cut, we use the relations (9.5) and (9.9):

$$\tau_{\theta z}^{(1)}(r, 0) = -\frac{2}{\pi} Mr + 2\mu \frac{\partial}{\partial r} \int_r^a \frac{g(t)dt}{\sqrt{t^2 - r^2}} \quad (r < a). \qquad (9.25)$$

Substituting (9.17) and (9.20) into (9.25), we can show that the shearing stresses $\tau_{\theta z}^{(1)}$ on the extension of the cut will have a singularity of the following type:

$$\tau_{\theta z}^{(1)}(r, 0) = -\frac{2\mu C(a)}{\sqrt{a^2 - r^2}} \sum_{m=0}^{\infty} P_{2m+1} r^{2m+1} + \ldots \quad (r < a). \qquad (9.26)$$

With the help of this equation, we can find the stress intensity factor at the tip of a cut:

$$K_{\text{III}} = \lim_{r \to a} [\sqrt{2\pi(a-r)}\, \tau_{\theta z}(r, 0)] = -2\mu \sqrt{\frac{\pi}{a}}\, g(a) \quad (r < a). \qquad (9.27)$$

Using the expansion (9.17), we can write

$$K_{\text{III}} = \frac{M}{4\sqrt{\pi a}} \left(\sum_{m=0}^{\infty} P_{2m+1} a^{2m+1} \right) \left(\sum_{m=0}^{\infty} P_{2m+1} \frac{a^{2m+3}}{2m+3} \right)^{-1}. \qquad (9.28)$$

In Fig. 63, the solid line shows the dependence of the quantity $M^* = \frac{3}{4} M\pi^{-1/2} R^{-5/2} K_{\text{III}C}^{-1}$ on the dimensionless radius ($\alpha = a/R$) of the tip of a crack (R is the radius of the cylinder, introduced in place of the unit radius).

It is clear that for small values of a (deep circular cut), $P_1 \sim 1$, $P_{2k+1} \sim 0$

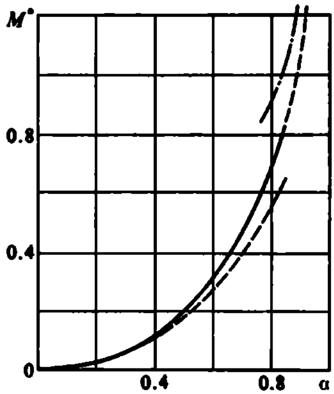

Fig. 63. Dependence of M^* on $\alpha = a/R$.

$(k = 1, 2, \ldots)$. It then follows from (9.28) that

$$K_{III} \sim \frac{3}{4} M\pi^{-1/2} a^{-5/2}. \tag{9.29}$$

This result (shown by a dotted line in Fig. 63) follows from the Neuber solution [61] of the problem of torsion of a rigid body of revolution having an external groove.

For shallow grooves on the surface of a cylinder, conversely, we can consider a half-plane with a cut extending to its boundary, under conditions of antiplane deformation (Fig. 64). The boundary conditions in this case can be written as follows:

$$\begin{gathered}\tau_{zx} = 0\,(x = 0), \quad u = 0\,(y = 0, \quad 1 - a \leqslant x), \\ \tau_{zy} = -\frac{2M}{\pi}(1 - x)\,(y = 0, \quad 0 \leqslant x \leqslant 1 - a).\end{gathered} \tag{9.30}$$

The second condition corresponds to the choice of stresses (with oppposite sign) emerging at a shallow groove upon torsion of a solid cylinder of unit radius.

In this case, we get

$$\Delta u = 0, \tau_{zy} = \mu \frac{\partial u}{\partial y}, \tau_{zx} = \mu \frac{\partial u}{\partial x}.$$

The components of displacements and stresses can be represented in the following form:

$$u(x, y) = \int_0^\infty A(\lambda) e^{-\lambda y} \cos \lambda x \, d\lambda,$$

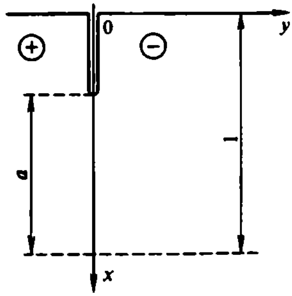

Fig. 64. A half-plane with a cut.

$$\tau_{zx} = -\mu \int_0^\infty \lambda A(\lambda) e^{-\lambda y} \sin \lambda x \, d\lambda,$$

$$\tau_{zy} = -\mu \int_0^\infty \lambda A(\lambda) e^{-\lambda y} \cos \lambda x \, d\lambda.$$

Satisfying the boundary conditions (9.30), we get, as a result of integration of τ_{zy} with respect to x:

$$\int_0^\infty A(\lambda) \sin \lambda x \, d\lambda = \frac{2M}{\pi\mu}\left(x - \frac{x^2}{2}\right) \quad (0 \leqslant x < 1 - a),$$

$$\int_0^\infty A(\lambda) \cos \lambda x \, d\lambda = 0 \quad (x > 1 - a).$$

Introducing a new unknown function $L(t)$, we get

$$A(\lambda) = \int_0^{1-a} L(t) J_0(\lambda t) \, dt.$$

This gives

$$\int_0^x \frac{L(t)dt}{\sqrt{x^2 - t^2}} = \frac{2M}{\pi\mu}\left(x - \frac{x^2}{2}\right), \quad L(t) = \frac{2M}{\pi\mu}\left(t - \frac{2}{\pi} t^2\right),$$

$$\tau_{zy}(x, 0) = -\mu \frac{\partial}{\partial x} \int_0^\infty A(\lambda) \sin \lambda x \, d\lambda$$

$$= -\mu \frac{\partial}{\partial x} \int_0^{1-a} L(t) \left[\int_0^\infty J_0(\lambda t) \sin \lambda x \, d\lambda \right] dt$$

$$= -\frac{2M}{\pi}(1 - x) - \delta\mu\frac{\partial}{\partial x}\int_0^{1-a}\frac{L(t)dt}{\sqrt{x^2 - t^2}}, \quad (9.31)$$

$$\delta = \begin{cases} 0, & 0 \leqslant x < 1 - a, \\ 1, & x > 1 - a \end{cases}$$

Taking (9.31) into account, the expression for the shearing stresses at the extension of a cut can be written as

$$\tau_{zy}(x, 0) = \frac{2Mx}{\pi\sqrt{x^2(1 - a)^2}}\left[1 - \frac{2(1 - a)}{\pi}\right] + \ldots \quad (x > 1 - a). \quad (9.32)$$

Here, we have neglected terms which do not tend to infinity as $x \to 1 - a$. The stress intensity factor can be written in the following form by taking (9.32) into consideration:

$$K_{\rm III} = \lim_{x \to 1 - a}\sqrt{2\pi[x - (1 - a)]}\tau_{zy}(x, 0)$$
$$= 2M\sqrt{\frac{1 - a}{\pi}}\left[1 - \frac{2(1 - a)}{\pi}\right]. \quad (9.33)$$

From this, we get for the limiting case

$$M^* = \frac{3M}{4\sqrt{\pi}K_{\rm IIIC}} = \frac{3}{8\sqrt{1 - \alpha}[1 - 2\pi^{-1}(1 - \alpha)]} \quad \left(\alpha = \frac{a}{R}\right). \quad (9.34)$$

The dot-and-dash line in Fig. 63 corresponds to the relation (9.34) (R is the characteristic length introduced in place of a unit length).

Combining the three solutions—the exact solution (9.27), the solution (9.29) for the case of a deep cut, and the solution (9.33) for a shallow groove, we can write the stress intensity factor in the following form: (τ_{max} is the maximum stress in the net cross section):

$$K_{\rm III} = \tau_{max}\sqrt{\pi R}F(\alpha), \quad \tau_{max} = \frac{2M}{\pi a^3}, \quad \alpha = \frac{a}{R}, \quad (9.35)$$

$$F(\alpha) = \frac{3}{8}\alpha^3(M^*)^{-1} \text{ (exact solution)},$$

$$F(\alpha) = F_N(\alpha) = \frac{3}{8}\alpha^{1/2} \quad \text{(deep cut)},$$

$$F(\alpha) = F_A(\alpha) = \alpha^3(1 - \alpha)^{1/2}\left[1 - \frac{2}{\pi}(1 - \alpha)\right] \text{ (shallow groove)}.$$

The values of $F(\alpha)$, $F_N(\alpha)$, and $F_A(\alpha)$ are given in Table 6.

TABLE 6

α		0.1	0.3	0.5	0.6	0.7
$F(\alpha)$	10^3	0.119	0.206	0.264	0.288	0.286
$F_N(\alpha)$	10^3	0.118	0.205	0.265	0.290	0.313
$F_A(\alpha)$	10^3	—	—	—	—	—
α		0.8	0.85	0.9	0.95	1
$F(\alpha)$	10^3	0.274	0.243	0.231	0.210	0
$F_N(\alpha)$	10^3	0.336	—	—	—	—
$F_A(\alpha)$	10^3	—	0.221	0.218	0.207	0

2. Let us consider the case of axial stretching of a cylinder of unit radius with a circular cut by a force $P = q\pi R^2$ (see Fig. 62). We shall find an approximate solution of this problem by assuming that the surface of the cut is free from load, while the shearing stresses and radial displacements on the lateral surface of the cylinder are equal to zero. The problem is axially symmetric, and the state of stress may be determined by considering a semi-infinite cylinder $z \geqslant 0$, for which the following conditions are satisfied at the end face:

$$\tau_{rz}(r,0) = 0 \quad (0 \leqslant r < a), \quad u_z(r,0) = 0 \quad (0 \leqslant r < a), \tag{9.36}$$
$$\sigma_z(r,0) = -q \quad (a < r \leqslant 1).$$

In this case, the components of displacements and stresses may be expressed in terms of one harmonic function from (5.45), (5.46), Ch. 3, Vol. 1, by putting $\psi = -(1-2\nu)^{-1}\varphi$ and $\partial\psi/\partial z = -\psi_z$:

$$u_r = z\frac{\partial^2\psi}{\partial r\,\partial z} + (1-2\nu)\frac{\partial\psi}{\partial r}, \quad u_z = z\frac{\partial^2\psi}{\partial z^2} - 2(1-\nu)\frac{\partial\psi}{\partial z},$$
$$\sigma_z = 2\mu\left(z\frac{\partial^3\psi}{\partial z^3} - \frac{\partial^2\psi}{\partial z^2}\right), \quad \tau_{zr} = 2\mu z\frac{\partial^3\psi}{\partial r\,\partial z^2}. \tag{9.37}$$

Taking into consideration the conditions at $z \to \infty$ and the axial symmetry, the harmonic function can be chosen in the form

$$\psi(r,t) = \sum_{n=1}^{\infty} \lambda_n^{-2}A_nJ_0(\lambda_n r)e^{-\lambda_n z}. \tag{9.38}$$

Here, λ_n are the roots of the equation $J_0'(\lambda_n) = 0$. In view of this, the following conditions must be satisfied on the lateral surface of the cylinder:

$$\tau_{rz}(1,z) = 0, \quad u_r = (1,z) = 0. \tag{9.39}$$

Satisfying the conditions (9.36), we get the following pair of series equations:

$$\sum_{n=1}^{\infty} \lambda_n^{-1} A_n J_0(\lambda_n r) = 0 \quad (0 \leqslant r < a),$$
$$\sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = \frac{q}{2\mu} \quad (a < r \leqslant 1). \tag{9.40}$$

In order to solve this system, we put

$$-\frac{q}{2\mu} + \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = -\frac{1}{r}\frac{\partial}{\partial r}\int_r^a \frac{t g(t)dt}{\sqrt{t^2 - r^2}} \quad (0 \leqslant r < a). \tag{9.41}$$

From the second equation in (9.40) and Eq. (9.41) it follows that

$$A_n = 2J_0^{-2}(\lambda_n)\int_0^a g(t)\cos(\lambda_n t)dt, \tag{9.42}$$

$$\int_0^a g(t)dt = -\frac{q}{4\mu}. \tag{9.43}$$

Substituting the expression (9.42) for the coefficients A_n into the first of Eqs. (9.40), and considering the dependences obtained in [60], we get

$$g(t) = \int_0^a g(u)K(u,t)du + \frac{4}{\pi}\int_0^a g(u)du, \tag{9.44}$$

$$K(u,t) = \frac{4}{\pi^2}\int_0^{\infty} \frac{K_1(y)}{yI_1(y)}[2I_1(y) - y\cosh(uy)\cosh(ty)]dy. \tag{9.45}$$

Thus, Eq. (9.44) is a Fredholm integral equation of the second kind, determining the function $g(t)$. In order to solve this equation, we use a power representation for the kernel of Eq. (9.45):

$$K(u,t) = \sum_{m=0}^{\infty} b_{2m}(u)t^{2m}. \tag{9.46}$$

Here,

$$b_0(u) = \frac{4}{\pi^2}\left[T^* - \sum_{s=1}^{\infty} T_{2s}\frac{u^{2s}}{(2s)!}\right],$$
$$b_{2m}(u) = -\frac{4}{\pi^2}\sum_{s=1}^{\infty}\frac{T_{2m+2,s-2}u^{2s-2}}{(2m)!(2s-2)!} \quad (m = 1, 2, \ldots), \tag{9.47}$$

$$T^* = \sum_{s=1}^{\infty} \frac{T_{2s}}{2^{2s} s!(s+1)!},$$

$$T_n = \int_0^{\infty} \frac{K_1(y)}{I_1(y)} y^n \, dy.$$

The numerical values of T_n are given in [62].

Seeking the solution of Eq. (9.44) in the form of the series

$$g(t) = C(a) \sum_{m=0}^{\infty} Q_{2m} t^{2m}, \tag{9.48}$$

we get an infinite system of algebraic equations for determining the coefficients Q_{2m}:

$$Q_{2m} = \sum_{k=0}^{\infty} Q_{2k} C_{2k,2m} + \delta_m^0 \quad (m = 0, 1, 2, \ldots). \tag{9.49}$$

Here,

$$C_{2k,0} = \frac{4}{\pi^2} T^* \frac{a^{2k+2}}{2k+1} - \frac{4}{\pi^2} \sum_{s=1}^{\infty} \frac{T_{2s} a^{2k+2s+1}}{(2s)!(2k+2s+1)}$$

$$(k = 0, 1, 2, \ldots),$$

$$C_{2k,2m} = -\frac{4}{\pi^2} \sum_{s=1}^{\infty} \frac{T_{2m+2s-2} a^{2k+2s-1}}{(2m)!(2s-2)!(2k+2s-1)}$$

$$(k = 0, 1, 2, \ldots; \; m = 1, 2, 3, \ldots).$$

The system (9.49) is quasi-regular for $0 \leqslant a < 1$. This follows from the asymptotic expressions for the functions $b_{2m}(u)$ for large values of m

$$|b_{2m}(u)| \sim \frac{2}{\pi(2-u)^{2m+1}} \left[1 + \frac{3}{4}\frac{2-u}{2m} + \frac{9}{32}\frac{(2-u)^2}{(2m-1)2m} + \ldots\right]$$
$$+ \frac{2}{\pi(2+u)^{2m+1}} \left[1 + \frac{3}{4}\frac{2+u}{2m} + \frac{9}{32}\frac{(2+u)^2}{(2m-1)2m} + \ldots\right]$$
$$(0 \leqslant u < 1)$$

and from the estimates

$$\left|C_{2k,2m}\right| \leqslant \left|b_{2m}(a_1)\right| \frac{a^{2k+1}}{2k+1} \quad (0 \leqslant a_1 < a < 1),$$

$$S_{2m+1} = \sum_{k=0}^{\infty} \left|C_{2k,2m}\right| \leqslant \frac{1}{2} \left|b_{2m}(a_1)\right| \ln \frac{1+a}{1-a}.$$

Thus, starting from a certain number $m = m'$, the following inequality is valid:

$$S_{2m+1} < 1, \quad m > m'.$$

Let us write down the asymptotic formula for determining the coefficients T_n for large values of n:

$$T_n \sim \frac{\pi n!}{2^{n+1}} \left[1 + \frac{3}{4}\frac{2}{n} + \frac{9}{32}\frac{2^2}{(n-1)n} + \dots \right].$$

For $n \geqslant 8$, the values of T_n calculated from this formula differ by no more than 1.1% from the exact values.

The constant $C(a)$ is defined in such a way that the condition (9.43) is satisfied. As a result, we get

$$C(a) = -\frac{q}{\pi G} \left(\sum_{m=0}^{\infty} Q_{2m} \frac{a^{2m+1}}{2m+1} \right)^{-1}. \tag{9.50}$$

The normal stresses in the plane $z = 0$ on the extension of a cut are given by the formula

$$\sigma_z(r, 0) = -2\mu \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) = -q + \frac{2\mu}{r}\frac{\partial}{\partial r} \int_r^a \frac{t g(t) dt}{\sqrt{t^2 - r^2}} \quad (r \leqslant a). \tag{9.51}$$

Substituting (9.48) into (9.51) and isolating the singularity for stresses at the tip of the cut, we get

$$\sigma_z(r, 0) = -\frac{2GC(a)}{\sqrt{a^2 - r^2}} \sum_{m=0}^{\infty} Q_{2m} r^{2m} + \dots \quad (r < a). \tag{9.52}$$

Here, we have neglected terms which are bounded for $r \to a$.

The formula (9.52) can be used to determine the stress intensity factor K_1, as well as the quantity Q_1 which is proportional to the critical load:

$$\begin{aligned} K_1 &= \sqrt{2\pi} \lim_{r \to a} [\sqrt{a - r}\, \sigma_z(r, 0)] = -2G \sqrt{\frac{\pi}{a}}\, g(a) \quad (r < a), \\ Q_1 &= 2qR^{1/2}\pi^{-1/2}K_{1C}^{-1}. \end{aligned} \tag{9.53}$$

Taking into consideration the expression (9.50) for the constant $C(a)$, we can write the expression for the stress intensity factor in the case of a cylinder of radius R:

$$K_1 = \frac{q\sqrt{\pi R}}{2\alpha^{3/2}} \left(\sum_{m=0}^{\infty} Q_{2m} \alpha^{2m} \right) \left(\sum_{m=0}^{\infty} Q_{2m} \frac{\alpha^{2m}}{2m+1} \right)^{-1} \quad \left(\alpha = \frac{a}{R} \right). \tag{9.54}$$

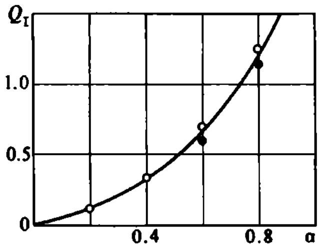

Fig. 65. Dependence of Q_I on $\alpha = a/R$.

For small values of α, we have $Q_0 \sim 1$, $Q_{2m} \sim 0$ ($m = 1, 2, \ldots$). It follows from (9.54) that

$$K_I \sim \frac{1}{2} q\sqrt{\pi R}\,\alpha^{-3/2}.$$

This result follows from Neuber's solution [61] of the problem of stretching of a space with an external slit.

Figure 65 shows the dependence of the quantity Q_I on the distance $\alpha = a/R$ from the tip of the cut. For the sake of comparison, the figure also contains Paris' data (light circles) and Buckner's data (dark circles) [63].

It is clear from the graph that the result obtained above occupies an intermediate position.

Section 10
The Problem of Electroelasticity for a Cylinder with an Electrode Coating

Let us consider the static problem of the electroelasticity for an infinitely long cylinder of radius a (Fig. 66), with an electrode coating on the region $r = a$, $-\theta_0 \leqslant \theta \leqslant \theta_0$ [64]. The cylinder material is either a transversely isotropic medium, like a crystal of hexagonal system, or a polarized piezoelectric ceramics whose direction of polarization coincides with the axis of the cylinder. This type of elements are contained in equipment for generating electroacoustic surface waves propagating in the azimuthal direction along the interface between the piezoelectric crystal ($r < a$) and vacuum ($r > a$).

Investigation of general equations for a piezoelectric medium shows that for crystals belonging to $6mm$ symmetry class, there exist coupled electroelastic shear waves. These waves are characterized by the fact that the mechanical displacement vector has a non-zero component u_z which is parallel to the axis of the cylinder and $u_z = u_z(r, \theta)$, while the non-zero components of the electric field intensity vector are $E_r(r, \theta)$ and $E_\theta(r, \theta)$.

Assuming that all the components of the electroelastic field are independent of the coordinate z and time t, we can write the basic system of equations, consisting of

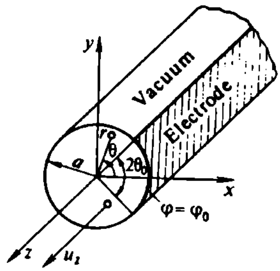

Fig. 66. A cylinder with an electrode coating.

equations of equilibrium, electrostatics, and the equations of state (5.17), Ch. 2, Vol. 1:

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial z} + \frac{1}{r}\tau_{rz} = 0, \tag{10.1}$$

$$\frac{\partial D_r}{\partial r} + \frac{1}{r}D_r + \frac{\partial D_\theta}{r\partial \theta} = 0, \quad \frac{\partial (rE_\theta)}{\partial r} = \frac{\partial E_r}{\partial \theta}, \tag{10.2}$$

$$\begin{gathered} \tau_{\theta z} = c_{44}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) - e_{15}E_\theta, \\ \tau_{rz} = c_{44}\frac{\partial u_z}{\partial r} - e_{15}E_r, \end{gathered} \tag{10.3}$$

$$D_r = e_{15}\frac{\partial u_z}{\partial r} + \varepsilon_{11}^s E_r, \quad D_\theta = e_{15}\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \varepsilon_{11}^s E_\theta.$$

Introducing the electric potential through the equations

$$E_r = -\frac{\partial \varphi}{\partial r}, \quad E_\theta = -\frac{\partial \varphi}{r\partial \theta} \tag{10.4}$$

and taking (10.3) into consideration, we obtain from (10.1) and (10.2)

$$\begin{gathered} c_{44}\nabla^2 u_z + e_{15}\nabla^2\varphi = 0, \quad e_{15}\nabla^2 u_z - \varepsilon_{11}^s\nabla^2\varphi = 0, \\ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \end{gathered} \tag{10.5}$$

Thus, in order to solve the static problem of electroelasticity, we must determine two harmonic functions u_z and φ in the domain $0 \leqslant r < a$ under the corresponding conditions at the boundary $r = a$.

Suppose that the deformation of a piezoelectric cylinder starts as a result of the action of an electric potential on an electrode situated on the surface $r = a$,

$-\theta_0 < \theta < \theta_0$ (see Fig. 66), the remaining part of the cylinder bordering with vacuum. If there are no mechanical loads on the surface $r = a$, and the electrode is considered as an infinitely thin conducting layer, the boundary conditions for the functions u_z and φ can be written in the following form after taking into account the symmetry with respect to x-axis:

$$\tau_{rz}\Big|_{r=a} = \left(c_{44}\frac{\partial u_z}{\partial r} + e_{15}\frac{\partial \varphi}{\partial r}\right)\Big|_{r=a} = 0 \quad (0 \leqslant \theta < \pi), \tag{10.6}$$

$$\varphi(a,\theta) = \varphi^*(a,\theta) \quad (\theta_0 \leqslant \theta < \pi), \tag{10.7}$$

$$\varphi(a,\theta) = \varphi_0 \quad (0 \leqslant \theta < \theta_0), \tag{10.8}$$

$$D_r(a,\theta) = D_r^*(a,\theta) \quad (\theta_0 < \theta < \pi). \tag{10.9}$$

Here, $\varphi^*(r, \theta)$ is the electric potential of vacuum, $D_r^*(r, \varphi) = -\varepsilon_0 \dfrac{\partial \varphi^*}{\partial r}$ is the radial component of the electric induction vector in vacuum, and ε_0 is the permittivity of vacuum $\left(\varepsilon_0 = \dfrac{10^{-9}}{36\pi}\,\dfrac{\Phi}{M}\right)$.

From the conditions (10.6) and (10.9), we get

$$(1 + k^2)\frac{\partial \varphi}{\partial r}\Big|_{r=a} = \frac{\varepsilon_0}{\varepsilon_{11}^S}\frac{\partial \varphi^*}{\partial r}\Big|_{r=a} \qquad (\theta_0 < \theta < \pi),$$

$$k^2 = \frac{e_{15}^2}{c_{44}\varepsilon_{11}^S}. \tag{10.10}$$

Thus, in this case the combined problem of electroelasticity is split into two parts: we can first determine the electric potentials φ and φ^* from the mixed conditions (10.7), (10.8), and (10.10), and then find the harmonic function u_z with the help of the condition (10.6).

The harmonic functions u_z, φ, and φ^* can be represented in the form of the series

$$u_z(r,\theta) = \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^n \cos n\theta \quad (0 \leqslant r < a), \tag{10.11}$$

$$\varphi(r,\theta) = \sum_{n=1}^{\infty} B_n \left(\frac{r}{a}\right)^n \cos n\theta \quad (0 \leqslant r < a), \tag{10.12}$$

$$\varphi^*(r,\theta) = D_0 \ln\left(\frac{r}{a}\right) + \sum_{n=1}^{\infty} C_n \left(\frac{a}{r}\right)^n \cos n\theta \quad (a < r < \infty). \tag{10.13}$$

It should be noted that the first term in the expansion (10.13) takes into account the unboundedness of the electrostatic potential of a plane problem at infinity [65].

Substituting the expansions (10.12) and (10.13) into the conditions (10.7), (10.8), and (10.10), we get the equality

$$C_n = B_n \tag{10.14}$$

and the pair of series equations of the type

$$\sum_{n=1}^{\infty} B_n \cos n\theta = \varphi_0 \quad (0 \leqslant \theta < \theta_0), \tag{10.15}$$

$$D_0^* + \sum_{n=1}^{\infty} nB_n \cos n\theta = 0 \qquad (\theta_0 < \theta < \pi),$$

$$D_0^* = \frac{D_0\varepsilon_0/\varepsilon_{11}^s}{1 + k^2 + \varepsilon_0/\varepsilon_{11}^s} \tag{10.16}$$

for determining the constants B_n.

Assuming that

$$D_0^* + \sum_{n=1}^{\infty} nB_n \cos n\theta = f(\theta) \qquad (0 \leqslant \theta < \theta_0), \tag{10.17}$$

we obtain, from Eqs. (10.16) and (10.17),

$$D_0^* = \frac{1}{\pi}\int_0^{\theta_0} f(\xi)d\xi, \qquad nB_n = \frac{2}{\pi}\int_0^{\theta_0} f(\xi)\cos n\xi\, d\xi. \tag{10.18}$$

Substituting (10.18) into (10.15) and changing the order of integration and summation, we get the following integral equation for the function $f(\theta)$:

$$\int_0^{\theta_0} f(\xi)\left(\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\theta \cos n\xi}{n}\right)d\xi = \varphi_0 \qquad (0 \leqslant \theta < \theta_0).$$

We can sum the kernel of this equation as follows:

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\theta \cos n\xi}{n} = -\frac{1}{\pi}\ln|2(\cos\theta - \cos\xi)|. \tag{10.19}$$

We then get

$$-\frac{1}{\pi}\int_0^{\theta} f(\xi)\ln|2(\cos\theta - \cos\xi)|\,d\xi = \varphi_0 \qquad (0 \leqslant \theta < \theta_0). \tag{10.20}$$

In order to solve Eq. (10.20), we introduce new variables ξ_1 and θ_1:

$$\cos\theta = \alpha + \beta\cos\theta_1, \quad \cos\xi = \alpha + \beta\cos\xi_1. \tag{10.21}$$

Here, $\alpha = \cos^2(\theta_0/2)$, $\beta = \sin^2(\theta_0/2)$, and the variables θ_1 and ξ_1 change between 0 and π as θ and ξ change between 0 and θ_0. Taking (10.21) into account, Eq. (10.20) can be written in the form

$$-\frac{1}{\pi}\left(\int_0^\pi \bar{f}(\xi_1)\frac{d\xi}{d\xi_1}d\xi_1\right)\ln 2\beta - \frac{1}{\pi}\int_0^\pi \bar{f}(\xi_1)\frac{d\xi}{d\xi_1}\ln|2(\cos\theta_1 - \cos\xi_1)|d\xi_1 = \varphi_0 \quad (10.22)$$

$$(0 \leqslant \theta_1 < \pi),$$

$$\bar{f}(\xi_1) = f[\arccos(\alpha + \beta\cos\xi_1)].$$

Expressing the function $\bar{f}(\xi_1)\dfrac{d\xi}{d\xi_1}$ in the form of the series,

$$\bar{f}(\xi_1)\frac{d\xi}{d\xi_1} = a_0 + \sum_{n=1}^{\infty} a_n \cos n\xi_1,$$

and using the expansion (10.19), we get from Eq. (10.22),

$$-a_0 \ln 2\beta = \varphi_0, \qquad a_n = 0 \qquad (n = 1, 2, \ldots). \qquad (10.23)$$

Consequently, the solution of the integral equation (10.20) can be written in the form

$$f(\xi) = a_0\frac{d\xi_1}{d\xi} = -\frac{\varphi_0}{\ln 2\beta}\,\frac{\sqrt{2}\cos\dfrac{\xi}{2}}{\sqrt{\cos\xi - \cos\theta_0}}. \qquad (10.24)$$

Substituting (10.24) into (10.18), we get

$$nB_n = -\frac{\varphi_0}{\pi\ln 2\beta}\,\frac{2\sqrt{2}}{\pi}\int_0^{\theta_0}\frac{\cos\dfrac{\xi}{2}\cos n\xi\, d\xi}{\sqrt{\cos\xi - \cos\theta_0}}$$

$$= -\frac{\varphi_0}{\pi\ln 2\beta}[P_{n-1}(\cos\theta_0) + P_n(\cos\theta_0)], \quad (10.25)$$

where $P_n(z)$ is the Legendre polynomial.

Thus, the electric potentials for a piezoelectric cylinder in vacuum will be given by the formulas

$$\varphi(r, \theta) = -\frac{\varphi_0}{\pi\ln 2\beta}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{P_{n-1}(\cos\theta_0) + P_n(\cos\theta_0)}{n}\cos n\theta, \quad (10.26)$$

$$\varphi^*(r,\theta) = -\frac{(1+k^2+\varepsilon_0/\varepsilon_{11}^s)}{\varepsilon_0/\varepsilon_{11}^s}\frac{\varphi_0}{\pi\ln 2\beta}\ln\left(\frac{r}{a}\right)$$

$$-\frac{\varphi_0}{\pi\ln 2\beta}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n\frac{P_{n-1}(\cos\theta_0)+P_n(\cos\theta_0)}{n}\cos n\theta. \tag{10.27}$$

The component u_z of the mechanical displacement vector can be obtained from (10.26) by taking into account the condition (10.6):

$$\left.\frac{\partial u_z}{\partial r}\right|_{r=a} = -\frac{e_{15}}{c_{44}}\left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = \frac{e_{15}\varphi_0}{c_{44}\pi\ln 2\beta}\frac{1}{a}\sum_{n=1}^{\infty}[P_{n-1}(\cos\theta_0)+P_n(\cos\theta_0)].$$

With the help of (10.11), we get the equations

$$A_n = \frac{e_{15}\varphi_0}{c_{44}\pi\ln 2\beta}\frac{P_{n-1}(\cos\theta_0)+P_n(\cos\theta_0)}{n}, \tag{10.28}$$

$$u_z(r,\theta) = \frac{e_{15}\varphi_0}{c_{44}\pi\ln 2\beta}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\frac{P_{n-1}(\cos\theta_0)+P_n(\cos\theta_0)}{n}\cos n\theta. \tag{10.29}$$

Formulas (10.26), (10.27), and (10.29) can be used to determine all the components of the electroelastic field of the given system. We shall mention another interesting point, according to which the stress components $\sigma_{\theta z}$ and σ_{rz} are equal to zero everywhere in the cylinder in view of the equality

$$u_z = -\frac{e_{15}}{c_{44}}\varphi,$$

while the following equations describe the components of the electric displacement vector:

$$D_r = -\varepsilon_{11}^s(1+k^2)\frac{\partial\varphi}{\partial r}$$

$$= \frac{\varepsilon_{11}^s\varphi_0(1+k^2)}{a\pi\ln 2\beta}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1}[P_{n-1}(\cos\theta_0)+P_n(\cos\theta_0)]\cos n\theta,$$

$$D_\theta = -\varepsilon_{11}^s(1+k^2)\frac{\partial\varphi}{r\partial\theta} \tag{10.30}$$

$$= -\frac{\varepsilon_{11}^s\varphi_0(1+k^2)}{a\pi\ln 2\beta}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{n-1}[P_{n-1}(\cos\theta_0)+P_n(\cos\theta_0)]\sin n\theta.$$

By using the expansions [66]

$$\sum_{n=1}^{\infty} [P_{n-1}(\cos\theta_0) + P_n(\cos\theta_0)]\cos n\theta = -1 + \begin{cases} \sqrt{2}\cos\dfrac{\theta}{2}(\cos\theta - \cos\theta_0)^{-1/2} & (\theta < \theta_0), \\ 0 & (\theta > \theta_0); \end{cases}$$

$$\sum_{n=1}^{\infty} [P_{n-1}(\cos\theta_0) + P_n(\cos\theta_0)]\sin n\theta = \begin{cases} \sqrt{2}\cos\dfrac{\theta}{2}(\cos\theta_0 - \cos\theta)^{-1/2} & (\theta > \theta_0), \\ 0 & (\theta < \theta_0), \end{cases}$$

the components D_r and D_θ on the surface of the cylinder can be written as follows:

$$D_r(a,\theta) = \frac{\varepsilon_{11}^s\varphi_0(1+k^2)}{a\pi\ln 2\beta}\left[-1 + \sqrt{2}\cos\frac{\theta}{2}(\cos\theta - \cos\theta_0)^{-1/2}\right] \quad (\theta < \theta_0),$$

$$D_\theta(a,\theta) = 0; \tag{10.31}$$

$$D_r(a,\theta) = -\frac{\varepsilon_{11}^s\varphi_0(1+k^2)}{a\pi\ln 2\beta} \quad (\theta > \theta_0),$$

$$D_\theta(a,\theta) = -\frac{\varepsilon_{11}^s\varphi_0(1+k^2)}{a\pi\ln 2\beta}\sqrt{2}\cos\frac{\theta}{2}(\cos\theta_0 - \cos\theta)^{-1/2}.$$

The expression for the electric charge density at the electrode is obtained with the help of Eq. (10.27):

$$q(\theta) = D_r^*(a,\theta) - D_r(a,\theta) = -\varepsilon_0\left.\frac{\partial\varphi^*}{\partial r}\right|_{r=a} - D_r(a,\theta)$$

$$= \frac{\varphi_0\varepsilon_{11}^s\left(1 + k^2 + \dfrac{\varepsilon_0}{\varepsilon_{11}^s}\right)}{a\pi\ln 2\beta}\left[2 - \sqrt{2}\cos\frac{\theta}{2}(\cos\theta - \cos\theta_0)^{-1/2}\right] \quad (0 \leqslant \theta < \theta_0). \tag{10.32}$$

Figure 67 shows the dependence of the quantity $q^*(\theta) = \dfrac{q(\theta)a\pi\ln 2\beta}{\varphi_0\varepsilon_{11}^s(1 + k^2 + \varepsilon_0/\varepsilon_{11}^s)}$ on the angle θ $\left(0 \leqslant \theta < \theta_0 = \dfrac{\pi}{4}\right)$. In order to calculate the sum in Eq. (10.29), we use the following integral representation:

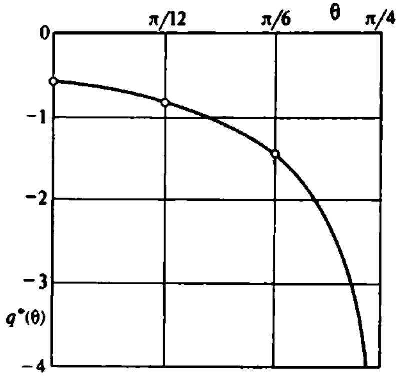

Fig. 67. Dependence of q^* on angle θ.

$$y(\cos\theta_0) = P_{n-1}(\cos\theta_0) + P_n(\cos\theta_0) = \frac{2\sqrt{2}}{\pi}\int_0^{\theta_0} \frac{\cos n\varphi \cos\frac{\varphi}{2}\,d\varphi}{\sqrt{\cos\varphi - \cos\theta_0}}. \qquad (10.33)$$

Then, after changing the order of summation and integration in (10.29), we get

$$u_z(r,\theta) = \frac{e_{15}\varphi_0}{c_{44}\pi\ln 2\beta}\,\frac{2\sqrt{2}}{\pi}$$

$$\times \int_0^{\theta_0} \frac{\cos\frac{\varphi}{2}}{\sqrt{\cos\varphi - \cos\theta_0}} \left(\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{\cos n\varphi \cos n\theta}{n} \right) d\varphi$$

$$= \frac{e_{15}\varphi_0\sqrt{2}}{c_{44}\pi^2\ln 2\beta} \int_0^{\theta_0} \frac{\cos\frac{\varphi}{2}}{\sqrt{\cos\varphi - \cos\theta_0}} \left(\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{\cos n(\varphi + \theta)}{n} \right.$$

$$\left. + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{\cos n(\varphi - \theta)}{n} \right) d\varphi.$$

We can sum the series in the last expression, since

$$\sum_{n=1}^{\infty} \frac{\rho^n \cos n\beta}{n} = -\frac{1}{2}\ln(1 + \rho^2 - 2\rho\cos\beta).$$

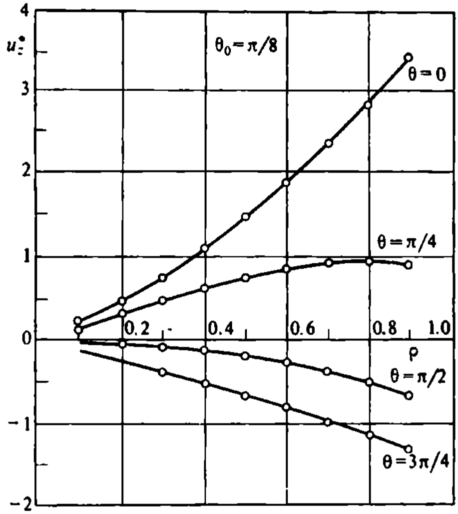

Fig. 68. Dependence of u_z^* on ρ and θ for $\theta_0 = \pi/8$.

Hence,

$$u_z(\rho, \theta) = -\frac{e_{15}\varphi_0\sqrt{2}}{2c_{44}\pi^2 \ln 2\beta} \int_0^{\theta_0} \frac{\cos\frac{\varphi}{2}}{\sqrt{\cos\varphi - \cos\theta_0}} \ln[(1 + \rho^2 - 2\rho\cos(\varphi + \theta)) \times (1 + \rho^2 - 2\rho\cos(\varphi - \theta))]d\varphi. \tag{10.34}$$

Introducing a change of variables

$$\gamma = \arcsin\left(\sin\frac{\varphi}{2}\Big/\sin\frac{\theta_0}{2}\right)$$

in the integral (10.34), we get the following representation for the displacement u_z:

$$u_z(\rho, \theta) = -\frac{e_{15}\varphi_0}{c_{44}\pi^2 \ln 2\beta}\int_0^{\pi/2} \ln\left\{\left[1 + \rho^2 - 2\rho\cos\theta\left(1 - 2\sin^2\frac{\theta_0}{2}\sin^2\gamma\right)\right]^2 - 16\rho^2\sin^2\theta\sin^2\frac{\theta_0}{2}\sin^2\gamma\left(1 - \sin^2\frac{\theta_0}{2}\sin^2\gamma\right)\right\}d\gamma. \tag{10.35}$$

The dependence of $u_z^* = -\dfrac{u_z(\rho,\theta)c_{44}\pi^2\ln 2\beta}{e_{15}\varphi_0}$ on $\rho = r/a$ and θ has been

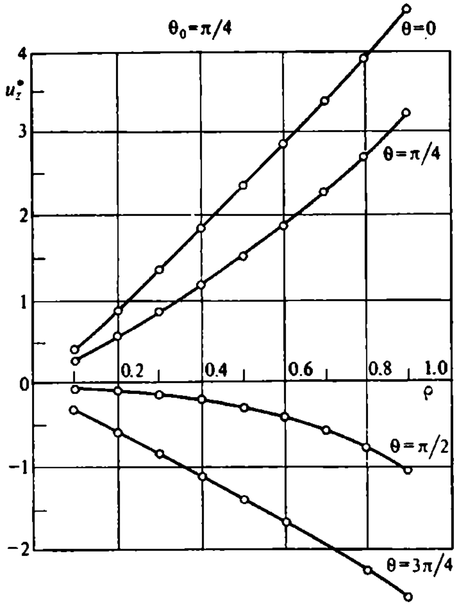

Fig. 69. Dependence of u_z^* on ρ and θ for $\theta = \pi/4$.

shown in Figs. 68 and 69 for two values $\theta_0 = \pi/8$ and $\theta_0 = \pi/4$ of the angle of electrode coating.

Section 11
Torsion Waves in a Space with a Crack in the Presence of an Axial Magnetic Field

Let us consider a circular cut of radius a, situated in an infinite homogeneous isotropic medium subjected to a homogeneous axial magnetic field $\mathbf{H}_0(0, 0, H_0)$ [67]. The medium has infinite conductivity and permeability of vacuum $\varkappa_0 = 4\pi\cdot 10^{-7}$ H/m (N/A^2). We introduce a cylindrical system of coordinates and direct the z-axis along the symmetry axis of the material. We shall consider small perturbations characterized by the displacement vector $\mathbf{u}[0, u_\theta(r, z, t), 0]$ and assume the perturbations to be independent of the angle θ. In this case, only the components $\tau_{r\theta}$ and $\tau_{\theta z}$ of the stress tensor are non-zero:

$$\tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta\right), \qquad \tau_{\theta z} = \mu\frac{\partial u_\theta}{\partial z}. \tag{11.1}$$

We put $\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$ and $\mathbf{E} = \mathbf{E}_0 + \mathbf{e}$, where $\mathbf{h}$ and $\mathbf{e}$ are perturbations in the magnetic and electric fields respectively. We then linearize the basic equations describing the motion in the presence of elastic electric and magnetic fields (see Sec. 5, Ch. 2, Vol. 1). In the MKS system, this linearized system of equations has the form [68]

$$e_r + \varkappa_0 H_0 \frac{\partial u_\theta}{\partial t} = 0, \tag{11.2}$$

$$\frac{\partial e_r}{\partial z} + \varkappa_0 \frac{\partial h_\theta}{\partial t} = 0, \tag{11.3}$$

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{1}{r^2} u_\theta + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{\varkappa_0 H_0}{\mu}\frac{\partial h_\theta}{\partial z} = \frac{1}{c_{\mathrm{T}}^2}\frac{\partial^2 u_\theta}{\partial t^2}. \tag{11.4}$$

Here, $\mathbf{e}(e_r, 0, 0)$ and $\mathbf{h}(0, h_\theta, 0)$ are perturbations arising in the electric and magnetic fields, and c_{T} is the velocity of the transverse waves. The components of displacement, stress, and perturbation of electric and magnetic fields can be represented in the following form:

$$\begin{aligned} u_\theta &= u_\theta^{(\mathrm{i})} + u_\theta^{(\mathrm{s})}, \\ e_r &= e_r^{(\mathrm{i})} + e_r^{(\mathrm{s})}, \\ h_\theta &= h_\theta^{(\mathrm{i})} + h_\theta^{(\mathrm{s})}, \end{aligned} \qquad \begin{aligned} \tau_{\theta z} &= \tau_{\theta z}^{(\mathrm{i})} + \tau_{\theta z}^{(\mathrm{s})}, \\ \tau_{r\theta} &= \tau_{r\theta}^{(\mathrm{i})} + \tau_{r\theta}^{(\mathrm{s})}. \end{aligned} \tag{11.5}$$

Here, the superscripts (i) and (s) refer to the incident and reflected waves respectively.

Suppose that the components of the incident wave are represented in the form

$$\begin{aligned} h_\theta^{(\mathrm{i})} &= \frac{ipH_0}{V_h c_{\mathrm{T}} a} u_\theta^0 r \exp\left[-ip\left(t + \frac{z}{V_h c_{\mathrm{T}}}\right)\right], \\ e_r^{(\mathrm{i})} &= \frac{ip\varkappa_0 H_0}{a} u_\theta^0 r \exp\left[-ip\left(t + \frac{z}{V_h c_{\mathrm{T}}}\right)\right], \\ u_\theta^{(\mathrm{i})} &= \frac{u_\theta^0}{a} r \exp\left[-ip\left(t + \frac{z}{V_h c_{\mathrm{T}}}\right)\right] \end{aligned} \tag{11.6}$$

$$(V_h = \sqrt{1 + h_c^2}, \quad h_c = \varkappa_0 H_0^2/\mu).$$

Here, u_θ^0 is the amplitude value of the incident elastic wave and p is the cyclic frequency.

The stresses in the incident wave are given by

$$\begin{gathered} \tau_{\theta z}^{(\mathrm{i})} = \mu \frac{\partial u_\theta^{(\mathrm{i})}}{\partial z} = P_h \frac{r}{a} e^{-ip\left(t + \frac{z}{V_h c_{\mathrm{T}}}\right)}, \\ P_h = -\frac{ip\mu}{V_h c_{\mathrm{T}}}. \end{gathered} \tag{11.7}$$

In order to solve this problem, we make use of Hankel's integral transformations, and the solution of Eqs. (11.2), (11.3), and (11.4) for the reflected wave can then be written for $z > 0$ in the following form:

$$u_\theta^{(s)}(r,z) = \int_0^\infty \alpha A(\alpha) e^{-\gamma(\alpha)z} J_1(\alpha r) d\alpha, \tag{11.8}$$

$$h_\theta^{(s)}(r,z) = -H_0 \int_0^\infty \alpha\gamma(\alpha) A(\alpha) e^{-\gamma(\alpha)z} J_1(\alpha r) d\alpha, \tag{11.9}$$

$$e_r^{(s)}(r,z) = -ip\varkappa_0 H_0 \int_0^\infty \alpha A(\alpha) e^{-\gamma(\alpha)z} J_1(\alpha r) d\alpha. \tag{11.10}$$

Here,

$$\gamma(\alpha) = \frac{1}{V_h}\left[\alpha^2 - \left(\frac{p}{c_T}\right)^2\right]^{1/2}, \quad \operatorname{Re}\gamma(\alpha) > 0, \quad \operatorname{Im}\gamma(\alpha) < 0.$$

Taking into account Eq. (11.8), we can find from (11.1) the stresses corresponding to the reflected wave:

$$\frac{1}{\mu}\tau_{r\theta}^{(s)} = \int_0^\infty \alpha^2 A(\alpha) e^{-\gamma(\alpha)z} J_0(\alpha r) d\alpha - \frac{2}{r}\int_0^\infty \alpha A(\alpha) e^{-\gamma(\alpha)z} J_1(\alpha r) d\alpha, \tag{11.11}$$

$$\frac{1}{\mu}\tau_{\theta z}^{(s)} = -\int_0^\infty \alpha\gamma(\alpha) A(\alpha) e^{-\gamma(\alpha)z} J_1(\alpha r) d\alpha. \tag{11.12}$$

The boundary conditions in the $z = 0$ plane have the form

$$\tau_{\theta z}\ (r, 0) = 0\,(r < a), \quad u_\theta^{(s)}(r, 0) = 0 \quad (r > a). \tag{11.13}$$

These conditions can be reduced with the help of Eqs. (11.12) and (11.8) to the following pair of integral equations for determining $A(\alpha)$:

$$\int_0^\infty \alpha^2 A(\alpha) J_1(\alpha r) d\alpha = \int_0^\infty \alpha g(\alpha) A(\alpha) J_1(\alpha r) d\alpha + \frac{V_h P_h}{\mu}\frac{r}{a} \quad (r < a), \tag{11.14}$$

$$\int_0^\infty \alpha A(\alpha) J_1(\alpha r) d\alpha = 0 \quad (r \geqslant a). \tag{11.15}$$

Multiplying Eq. (11.14) by r^2 and integrating with respect to r between 0 and r, we get

$$\int_0^\infty \alpha A(\alpha)J_2(\alpha r)d\alpha = \int_0^\infty g(\alpha)A(\alpha)J_2(\alpha r)d\alpha + \frac{V_h P_h}{\mu}\frac{r^2}{4a} \qquad (r < a). \tag{11.16}$$

The solution of the system (11.15), (11.16) will be sought in the form

$$\alpha A(\alpha) = \sqrt{\alpha} \int_0^a \varphi_h(\xi)J_{3/2}(\alpha\xi)d\xi, \tag{11.17}$$

where $\varphi_h(\xi)$ is a certain unknown function which is yet to be determined. Substituting (11.17) into (11.15) and changing the order of integration, we get

$$\int_0^\infty \alpha A(\alpha)J_1(\alpha r)d\alpha$$

$$= \int_0^a \varphi_h(\xi)\left(\int_0^\infty \sqrt{\alpha}J_1(\alpha r)J_{3/2}(\alpha\xi)d\alpha\right)d\xi. \tag{11.18}$$

The inner integral in (11.18) can be determined as follows:

$$\int_0^\infty \sqrt{\alpha}J_1(\alpha r)J_{3/2}(\alpha\xi)d\alpha = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{\pi}}\dfrac{r}{\xi^{3/2}}(\xi^2 - r^2)^{-1/2} & (r < \xi), \\ 0 & (r > \xi). \end{cases}$$

Then the equality (11.15) will be identically satisfied for $r > a$.

Substituting Eq. (11.17) into (11.16) and taking into account the value of the integral

$$\int_0^\infty \sqrt{\alpha}J_2(\alpha r)J_{3/2}(\alpha\xi)d\alpha = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{\pi}}\dfrac{\xi^{3/2}}{r^2}(r^2 - \xi^2)^{-1/2} & (\xi < r), \\ 0 & (\xi > r), \end{cases}$$

we can rewrite Eq. (11.16) in the form

$$\sqrt{\frac{2}{\pi}}\int_0^r \frac{\varphi_h(\xi)\xi^{3/2}d\xi}{\sqrt{r^2 - \xi^2}} = r^2 \int_0^a \varphi_h(\eta)\left[\int_0^\infty \alpha^{-1/2}g(\alpha)J_2(\alpha r)J_{3/2}(\alpha\eta)d\alpha\right]d\eta$$

$$+ \frac{V_h P_h}{\mu}\frac{r^4}{4a} \qquad (r < a). \tag{11.19}$$

Assuming that the right-hand side of this equation is known, we can solve it in $\varphi_h(\xi)$. This gives a Fredholm integral equation of the second kind:

$$\varphi_h(\xi) = \int_0^a \varphi_h(\eta)\left(\int_0^\infty g(\alpha)\xi J_{3/2}(\alpha\xi)J_{3/2}(\alpha\eta)d\alpha\right)d\eta + \frac{2\sqrt{2}V_hP_h\xi^{5/2}}{\sqrt{\pi}\,3\mu a}.$$

Introducing the dimensionless quantities

$$s = \frac{\xi}{a}, \qquad t = \frac{\eta}{a},$$
$$\Phi_h(s) = \frac{3\sqrt{\pi}\mu}{2\sqrt{2}a\sqrt{a}P_h\sqrt{s}}\varphi_h(as), \tag{11.20}$$

we can transform the Fredholm equation as follows:

$$\Phi_h(s) = \int_0^1 \Phi_h(t)K_h(s,t)dt + s^2. \tag{11.21}$$

Here,

$$K_h(s,t) = -a^2\sqrt{st}\int_0^\infty (\alpha - \sqrt{\alpha^2 - (p/c_T)^2})J_{3/2}(\alpha as)J_{3/2}(\alpha at)d\alpha. \tag{11.22}$$

Thus, the problem is reduced to the solution of the integral equation (11.21) and to the determination of the function $\varphi_h(\xi)$ with the help of which we can find all the components of elastic and electromagnetic fields for the reflected wave.

It should be noted that the kernel (11.22) of the integral equation (11.21) can be transformed with the help of integration along the contour [69] and expressed as an integral with finite limits:

$$K_h(s,t) = -iP_h^2\sqrt{st}\int_0^1 \sqrt{1-\xi^2}J_{3/2}(P_hs\xi)H_{3/2}^{(1)}(P_ht\xi)d\xi \qquad (s < t). \tag{11.23}$$

Here, $H_{3/2}^{(1)}(z)$ is the Hankel function. For $s > t$, the expression for the kernel can be obtained by interchanging s and t.

Let us find the asymptotic expressions for stresses in the vicinity of the crack contour $r = a$, $z = 0$. Naturally, the stresses obtained are the sum of the dynamic elastic stresses $\tau_{r\theta}$, $\tau_{\theta z}$ and the dynamic Maxwellian stresses $\tau_{mr\theta}$ and $\tau_{m\theta z}$. Moreover, it is clear from (11.6) and (11.7) that the stresses $\tau_{r\theta}^{(i)}$ and $\tau_{\theta z}^{(i)}$, corresponding to the incident wave, have no singularities for $r \to a$, $z \to 0$.

Let us find the asymptotic expression for $\tau_{\theta z}^{(s)}$ as $r \to a$, $z \to 0$. Integrating by parts, we can write the expression (11.17) in the form

$$\alpha A(\alpha) = -\alpha^{-1/2}\varphi_h(a)J_{1/2}(\alpha a) + \alpha^{-1/2}\int_0^a \xi^{-1/2}(\varphi_h(\xi)\xi^{1/2})' J_{1/2}(\alpha\xi)d\xi. \tag{11.24}$$

Substituting (11.24) into (11.12), we get

$$\frac{1}{\mu}\tau_{\theta z}^{(s)} = \varphi_h(a)\int_0^\infty \gamma(\alpha)\alpha^{-1/2}e^{-\gamma(\alpha)z}J_{1/2}(\alpha a)J_1(\alpha r)d\alpha$$

$$-\int_0^a \xi^{-1/2}(\varphi_h(\xi)\xi^{1/2})'\left[\int_0^\infty \gamma(\alpha)\alpha^{-1/2}e^{-\gamma(\alpha)z}J_{1/2}(\alpha\xi)J_1(\alpha r)d\alpha\right]d\xi. \qquad (11.25)$$

It can be shown [58] that the singular part $\tau_{\theta z}^{(s)}$ of the stresses is determined by the first term in (11.25), and the following approximate equation can be used for evaluating the integral:

$$\gamma(\alpha) \sim \frac{\alpha}{V_h} \quad \text{for} \quad \alpha \to \infty. \qquad (11.26)$$

Taking this into account, we get

$$\frac{1}{\mu}\tau_{\theta z}^{(s)} \sim \frac{\varphi_h(a)}{V_h}\sqrt{\frac{2}{\pi a}}\int_0^\infty e^{-\frac{z\alpha}{V_h}}J_1(\alpha r)\sin\alpha a\, d\alpha. \qquad (11.27)$$

The value of the integral in (11.27) is obtained on the basis of the well-knowr equality [4]

$$\int_0^\infty e^{-\alpha z}J_1(\alpha r)d\alpha = \frac{1}{r} - \frac{z}{r\sqrt{z^2+r^2}} \quad (\mathrm{Re}\,\alpha > 0).$$

Isolating the singular part of the integral in (11.27) for $r \to a$, $z \to 0$, we get

$$\tau_{\theta z}^{(s)} \sim \frac{\mu\varphi_h(a)}{V_h a\sqrt{\pi r_1}}H\left(\frac{1}{V_h},\theta\right), \qquad (11.28)$$

where

$$r_1\cos\theta = r - a, \quad z\sin\theta = r_1,$$

$$H\left(\frac{1}{V_h},\theta\right) = \left(\frac{\sqrt{\cos^2\theta + (1/V_h^2)\sin^2\theta} + \cos\theta}{2(\cos^2\theta + (1/V_h^2)\sin^2\theta)}\right)^{1/2}.$$

With the help of (11.28), we can find the stress intensity factor

$$K_{h\text{III}} = \lim_{r\to a}\sqrt{2(r-a)}\,\tau_{\theta z}^{(s)} = \frac{4}{3\pi}V_h^2 P_h\sqrt{a}\,\Phi_h(1). \qquad (11.29)$$

Then,

$$\tau_{\theta z}^{(s)} \sim \frac{K_{h\text{III}}}{V_h^2\sqrt{2r_1}}H\left(\frac{1}{V_h},\theta\right). \qquad (11.30)$$

The asymptotic expressions for $\tau_{r\theta}^{(s)}$, $\tau_{mr\theta}^{(s)}$, and $\tau_{m\theta z}^{(s)}$ are obtained in a similar manner.

It should be noted that the solution obtained for $h \to 0$ is identical to the solution [70] for purely elastic case.

Section 12
The Dynamic Problem of Thermoelasticity

Following [71], let us use the Laplace transformation (see Sec. 4, Ch. 1, Vol. 1) to consider the problem of determining the stresses in a half-space of the one-dimensional case. We shall assume that the temperature $T(x, t)$ depends only on the time t and the coordinate x (the x-axis is perpendicular to the boundary of the half-space). Naturally, the stresses and displacements arising as a result of the inhomogeneity of the temperature field will also depend only on the coordinate x and time t. Moreover, the displacements v and w, as well as the tangential components τ_{xy} and τ_{xz}, become identically equal to zero. In this case, two of the three equations of motion (1.11) of Ch. 2, Vol. 1 are identically satisfied and we are left only with the equation

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}. \tag{12.1}$$

Differentiating this equation with respect to x and substituting the Hooke's law expression for the strain $\dfrac{\partial u}{\partial x}$, we obtain the equation

$$a^2 \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_x}{\partial t^2} = s \frac{\partial^2 T}{\partial t^2}, \tag{12.2}$$

where $a = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ (the velocity of propagation of the expansion waves), $s = \alpha(2\mu + 3\lambda)$, and α is the coefficient of linear expansion.

We shall solve the problem of an abrupt heating of the boundary of a half-space. The temperature distribution satisfies the thermal conductivity equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad (x \geqslant 0, t \geqslant 0), \tag{12.3}$$

the initial condition

$$T(x, t)|_{t=0} = 0, \tag{12.4}$$

and the boundary condition

$$T(x, t)|_{x=0} = T_0. \tag{12.5}$$

In order to solve the problem formulated above, we perform the Laplace transformation in time. The ordinary differential equation for the transform $\bar{T}(x, p)$ can be easily solved:

$$\bar{T}(x, p) = \frac{T_0}{p} \exp\left(-x\sqrt{\frac{p}{k}}\right). \tag{12.6}$$

For the stresses σ_x we have the initial conditions

$$\sigma_x\Big|_{t=0} = \frac{\partial \sigma_x}{\partial t}\Big|_{t=0} = 0 \tag{12.7}$$

and the boundary condition

$$\sigma_x|_{x=0} = 0. \tag{12.8}$$

The limiting condition $|\sigma_x| < \infty$ must also be satisfied for $x \to \infty$.

Multiplying both sides of Eq. (12.2) by e^{-pt} and integrating with respect to t between 0 and ∞, while taking into account the initial conditions for $\sigma_x(x, t)$ and $T(x, t)$, we get the following equation for the transform:

$$a^2 \frac{\partial^2 \bar{\sigma}_x}{\partial x^2} - p^2 \bar{\sigma}_x = spT_0 \exp\left(-x\sqrt{\frac{p}{k}}\right). \tag{12.9}$$

We require that the boundary condition $\bar{\sigma}_x|_{x=0} = 0$ be satisfied in addition to the condition of boundedness of $\bar{\sigma}_x$ for $x \to \infty$. The solution of Eq. (12.9) then assumes the form

$$\bar{\sigma}_x = \frac{sT_0}{p - a^2/k} \exp\left(-\frac{px}{a}\right) - \frac{sT_0}{p - a^2/k} \exp\left(-x\sqrt{\frac{p}{k}}\right). \tag{12.10}$$

Reverting to the original expression, we obtain the following expressions for the stresses:

$$\sigma_x = -sT_0 \int_0^t \exp\left[\frac{a^2}{k}(t-\tau)\right] \frac{x}{2\sqrt{\pi k \tau^3}} \exp\left(-\frac{x^2}{4k\tau}\right) d\tau, \quad t < \frac{x}{a},$$

$$\sigma_x = sT_0 \left\{\exp\left[\frac{a^2}{k}\left(t - \frac{x}{a}\right)\right] - \int_0^t \exp\left[\frac{a^2}{k}(t-\tau)\right] \frac{x}{2\sqrt{\pi k \tau^3}} \right.$$
$$\left. \times \exp\left(-\frac{x^2}{4k\tau}\right) d\tau \right\}, \quad t > \frac{x}{a}. \tag{12.11}$$

Chapter Seven

The Methods of Potential in the Theory of Elasticity

Section 1
Generalized Elastic Potentials

Let us suppose that a concentrated force $\varphi(\varphi_1(q), \varphi_2(q), \varphi_3(q))$ is applied at a certain point $q(y_1, y_2, y_3)$ in space. Then, using the formulas (5.27), Ch. 3, Vol. 1, as well as the formulas obtained by cyclic permutation, we get the expressions for the displacements at an arbitrary point $p(x_1, x_2, x_3)$. These expressions can be conveniently written in a compact form as the product of a certain matrix $\mathbf{\Gamma}(p, q)$, called the Kelvin-Somilyana matrix, and the vector $\varphi(q)$:

$$\mathbf{u}(p) = \mathbf{\Gamma}(p, q)\varphi(q). \tag{1.1}$$

The expressions for the matrix elements of $\mathbf{\Gamma}(p, q)$[1] are:

$$\mathbf{\Gamma}_{ij} = \frac{1}{4\pi(\lambda + 2\mu)\mu}\left[(\lambda + \mu)\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} + (\lambda + 3\mu)\delta_{ij}\right]\frac{1}{r(p, q)}.$$

Let us now suppose that at a certain closed Lyapunov surface S in space, the forces $\varphi(q)$ are given. Then the integral

$$\mathbf{V}(p) = \int_S \mathbf{\Gamma}(p, q)\varphi(q)\, dS_q \tag{1.2}$$

will be a vector function satisfying Lamé's equations in the entire space except the surface S. In analogy with the harmonic potential, this function is called the generalized elastic potential of a single layer. Further, we assume that the function $\varphi(q)$, also called the density, belongs to the H-L class.

We shall mention certain properties of the potential $\mathbf{V}$, which are just similar to the properties of the potential of a single layer. The potential (1.2) may be determined directly at points on the surface (layer carrier), and its limiting values (from inside and outside) are identical and equal to the proper value. Consequently, the generalized elastic potential of a single layer is a vector function which is continuous everywhere in space. It should be noted that the potential $\mathbf{V}(p)$ at infinity tends to zero as $1/R$.

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[1] For the sake of simplicity, the factor 2 has been introduced into the subsequent formulas.

Let us now define a certain point p in space, and a small area element with ν as the direction of the normal. Then, the stress vector acting on this area element may be written in the form of the following integral with the help of the expression (3.37), Ch. 2, Vol. 1, for the stress operator:

$$T_\nu \mathbf{V}(p) = \int_S \mathbf{\Gamma}_1(p, q)\varphi(q)\, dS_q, \tag{1.3}$$

where $\mathbf{\Gamma}_1(p, q)$ is a certain matrix whose elements are given by

$$\Gamma_{1ij} = -\left[m\delta_{ij} + n\frac{(x_i - y_i)(x_j - y_j)}{r^2}\right]\frac{\sum_{l=1}^{3}(x_l - y_l)\nu_{l(p)}}{r^3} + m\left[\nu_{i(p)}\frac{(x_j - y_j)}{r^3} - \nu_{j(p)}\frac{(x_i - y_i)}{r^3}\right] \tag{1.4}$$

$$\left(m = \frac{1}{2\pi}\frac{\mu}{\lambda + 2\mu},\ n = \frac{3}{2\pi}\frac{\lambda + \mu}{\lambda + 2\mu}\right).$$

In view of the regularity of the integrand in (1.3), the order of differentiation and integration can be interchanged at all points in the space, with the exception of the surface S.

Let us now consider the points lying in close proximity of the surface S so that the direction ν perpendicular to the surface can be uniquely determined at each of these points. We shall now consider the stress vector generated by the single layer potential and the above-mentioned directions of the normal. Taking this into account, we can rewrite the integral (1.3):

$$T_{\nu(q_1)}\mathbf{V}(p) = \int_S \mathbf{\Gamma}_1(p, q)\varphi(q)dS_q. \tag{1.3'}$$

The point q_1 is such a point on the surface S that the normal erected at this point passes through the point p.

Let us now perform the limit transition (from inside as well as outside) for the stress vector to the points on the surface:

$$\lim_{p \to q_1 (p \in D^-)} T_{\nu(q_1)}\mathbf{V}(p) = \lim \int_S \mathbf{\Gamma}_1(p, q)\varphi(q)dS_q = T_\nu^- \mathbf{V}(q_1), \tag{1.5}$$

$$\lim_{p \to q_1 (p \in D^+)} T_{\nu(q_1)}\mathbf{V}(p) = \lim \int_S \mathbf{\Gamma}_1(p, q)\varphi(q)dS_q = T_\nu^+ \mathbf{V}(q_1). \tag{1.6}$$

The proper value of the integral

$$\int_S \boldsymbol{\Gamma}_1(p, q)\varphi(q)dS_q$$

cannot be considered in the ordinary sense (as an improper integral) since the last terms in the non-diagonal elements (1.4) have a second-order singularity and hence this integral must be considered in the sense of the principal value (see Sec. 3, Ch. 1, Vol. 1). It can be easily shown that in this case the conditions (3.20), Ch. 1, Vol. 1, necessary for the existence of a singular integral, are satisfied.

Let us also notice the analogy with the harmonic potential of a single layer. The limiting values of the stress operator are different and do not coincide with the proper value. The corresponding dependences will be derived at a later stage.

Let us now return to the matrix $\boldsymbol{\Gamma}_1(p, q)$ and construct the matrix $\boldsymbol{\Gamma}_2(p, q)$ conjugate to it ($\Gamma_{2ij}(p, q) = \Gamma_{1ij}(q, p)$). In other words, we construct a matrix obtained by interchanging the points p and q and by transposing the elements. The elements of this matrix are given by

$$\Gamma_{2ij}(p, q) = \left[m\delta_{ij} + n\frac{(y_i - x_i)(y_j - x_j)}{r^2}\right] \sum_{l=1}^{3} \frac{(x_l - y_l)\nu_l(q)}{r^3} + m\left[\nu_i(q)\frac{x_j - y_j}{r^3} - \nu_j(q)\frac{x_i - y_i}{r^3}\right]. \quad (1.7)$$

It can be easily shown that each column of the matrix $\boldsymbol{\Gamma}_2(p, q)$ satisfies Lamé's equations in the variable p. Hence the product $\boldsymbol{\Gamma}_2(p, q)\varphi(q)$ ($\varphi(q)$ is an arbitrary vector) will satisfy these equations everywhere in space with the exception of the point q. The vector thus constructed may be treated as a displacement field generated in the entire space by the concentrated moment $\varphi(q)$ applied at the point q in the plane with the normal ν. We then construct the integral

$$\mathbf{W}(p) = \int_S \boldsymbol{\Gamma}_2(p, q)\varphi(q)dS_q, \quad (1.8)$$

where S is the Lyapunov surface as before. This integral is called the generalized elastic potential of a double layer[2]. It follows from the formulas (1.7) that the proper value of the generalized elastic potential of a double layer is a singular integral. Later, we shall establish the relations between the limiting and proper values of the double layer potential (assuming that the density belongs to H-L class).

In order to prove the above statements, we must make a number of constructions. Let us consider Betti's second formula (4.26), Ch. 2, Vol. 1, and apply it to the region enclosed between the given surface S and a sphere σ_ε of radius ε having its centre at a certain point p_0. Suppose that the displacement $\mathbf{u}(p)$ in this case satisfies

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[2] In literature [72] sometimes the words "of the first kind" are added. But since we shall not be considering the double layer potential of the second kind in this book, we shall confine ourselves to the shortened definition. These potentials were introduced in [73].

Lamé's equation over the entire domain D^+, and that the displacement $\mathbf{v}(p)$ is generated by a force applied at the point p_0, one of whose components (say, ith) is equal to unity, and all the remaining components are equal to zero. Henceforth, we shall denote these forces by $\mathbf{a}_i$. With the help of the notation introduced above, we get

$$\mathbf{v}_i(p) = \Gamma(p, p_0)\mathbf{a}_i.$$

We start the constructions with $i = 1$. Betti's formula in this case assumes the form

$$\int\limits_{D^+ \cap |p-p_0| > \varepsilon} (\mathbf{u}\Delta^*\mathbf{v}_1 - \mathbf{v}_1\Delta^*\mathbf{u})d\Omega = \int\limits_{S+\sigma_\varepsilon} (\mathbf{u}T_\nu\mathbf{v}_1 - \mathbf{v}_1T_\nu\mathbf{u})dS = 0. \quad (1.9)$$

The volume integrals vanish, since the displacements $\mathbf{u}$ and $\mathbf{v}$ satisfy Lamé's equations in the domain under consideration. Let us consider the integral

$$\int\limits_{\sigma_\varepsilon} \mathbf{v}_1 T_\nu \mathbf{u} dS.$$

Since the displacement $\mathbf{v}_1(p)$ has a first-order pole at the point p_0, and the stresses $T_\nu\mathbf{u}$ are bounded, the surface integral of the second term must tend to zero for $\varepsilon \to 0$ and can therefore be excluded from further consideration. In view of the continuity of the displacement $\mathbf{u}(p)$ in the neighbourhood of the point p_0, the integral

$$\int\limits_{\sigma_\varepsilon} \mathbf{u}T_\nu\mathbf{v}_1 dS$$

(in the limit) must coincide with the integral

$$\mathbf{u}(p_0)\int\limits_{\sigma_\varepsilon} T_\nu\mathbf{v}_1 dS. \quad (1.10)$$

Using the spherical system of coordinates, let us calculate this integral. The following equalities may be easily obtained:

$$\int\limits_{\sigma_\varepsilon} \frac{\partial}{\partial\nu}\frac{1}{r(p, p_0)} dS_p = -4\pi$$

$$\int\limits_{\sigma_\varepsilon} \left(\frac{\partial r}{\partial x_i}\right)\left(\frac{\partial r}{\partial x_j}\right)\frac{\partial}{\partial\nu}\frac{1}{r(p, q_0)} dS_p = 0 \qquad (i \neq j),$$

$$\int\limits_{\sigma_\varepsilon} \left(\frac{\partial r}{\partial x_i}\right)^2\frac{\partial}{\partial\nu}\frac{1}{r(p, p_0)} dS_p = -\frac{4}{3}\pi.$$

From parity considerations it follows that

$$\int\limits_{\sigma_\varepsilon} \left[\nu_i(p)\frac{x_j - y_j}{r^3} - \nu_j(p)\frac{x_i - y_i}{r^3}\right] dS_p = 0.$$

Having obtained in this way the values of all the integrals appearing in (1.10), we get in the limit the representation for the component $u_1(p)$:

$$2u_1(p_0) = -\int_S T_{\nu(p)}\Gamma(p_0, p)\mathbf{a}_1\mathbf{u}(p)dS_p + \int_S \Gamma(p_0, p)\mathbf{a}_1 T_{\nu(p)}\mathbf{u}(p)dS_p. \quad (1.11)$$

Similar formulas are obtained by using the displacements $\mathbf{v}_2$ and $\mathbf{v}_3$. Adding these formulas, we can write the final result in the following form:

$$2\mathbf{u}(p) = -\int_S \Gamma_2(p, q)\mathbf{u}(q)dS_q + \int_S \Gamma(p, q)T_\nu\mathbf{u}(q)dS_q. \quad (1.12)$$

Let us now suppose that the point p is chosen outside the surface S. This leads to the equality

$$0 = -\int_S \Gamma_2(p, q)\mathbf{u}(q)dS_q + \int_S \Gamma(p, q)\, T_\nu\mathbf{u}(q)dS_q. \quad (1.13)$$

By carrying out similar constructions, when the displacement $\mathbf{u}(p)$ has been determined for the domain D^- which is external with respect to the surface S, we get the formulas

$$2\mathbf{u}(p) = \int_S \Gamma_2(p, q)\mathbf{u}(q)dS_q - \int_S \Gamma(p, q)\, T_\nu\mathbf{u}(q)dS_q \quad (p \in D^-), \quad (1.14)$$

$$0 = \int_S \Gamma_2(p, q)\mathbf{u}(q)dS_q - \int_S \Gamma(p, q)T_\nu\mathbf{u}(q)dS_q \quad (p \in D^+). \quad (1.15)$$

Since Betti's formulas are valid for domains bounded by several surfaces, it is apparent that the identities obtained above also remain valid.[3]

Let us once again consider the limiting properties for displacements generated by a double layer potential, and for stresses generated by a single layer potential.

We start with a consideration of the first question. We specify in the domain D^+ a certain vector function of constant magnitude φ_0 and treat it as the displacement of the entire domain D^+. We then get the following equality from (1.12):

$$2\varphi_0 = -\int_S \Gamma_2(p, q)\varphi_0 dS_q, \quad (1.16)$$

where the point p has been taken in D^+. If, however, this point lies in the domain D^-, we get, in accordance with (1.13),

$$0 = \int_S \Gamma_2(p, q)\varphi_0 dS_q. \quad (1.17)$$

Let us now go over to the computation of the singular (proper) value of the in-

[3] It should be noted that the identities (1.12)-(1.15) do not give a solution of the boundary value problem, since by definition either the displacements or the stresses are specified on the boundary surface. Nevertheless, these will be used later while constructing the solutions.

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$$\int_S \mathbf{\Gamma}_2(p, q)\varphi_0 dS_q,$$

where the point p lies on the surface S. We choose a certain point q_1 on the surface S; and taking this point as the centre, we form a sphere of radius ε. We denote by σ_ε^+ and $\sigma_\varepsilon^-$the two parts of this sphere, lying in the domains D^+ and D^- respectively. The part of the surface S contained outside this sphere is denoted by S_ε.

Let us apply the identity (1.16) to the domain bounded by the surface $S_\varepsilon \cup \sigma_\varepsilon^+$. The integral is found to be equal to $-2\varphi_0$. If, however, we take a domain bounded by the surface $S_\varepsilon \cup \sigma_\varepsilon$, the integral is found to be equal to zero. Taking into account the smoothness of the surface S (in this case, the local smoothness in the vicinity of the point q_1 is important), we find that in the limit $\varepsilon \to 0$, the integrals over $\sigma_\varepsilon^+$and $\sigma_\varepsilon^-$are identical (to within the sign). Thus, in the limit, we get the equality

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} \mathbf{\Gamma}_2(q_1, q)\varphi_0 dS_q = \int_S \mathbf{\Gamma}_2(q_1, q)\varphi_0 dS_q = -\varphi_0. \tag{1.18}$$

It should be recalled that it is just through the domain S_ε that we determined the two-dimensional singular integral (see Sec. 3, Ch. 1, Vol. 1).

The above description forms the basis of the generalized Gaussian theorem. In another form, this theorem can be written as follows:

$$\begin{aligned} &\int_S \mathbf{\Gamma}_2(p, q) dS_q = -2\mathbf{E} \quad (p \in D^+),\\ &\int_S \mathbf{\Gamma}_2(p, q) dS_q = \mathbf{0} \quad (p \in D^-),\\ &\int_S \mathbf{\Gamma}_2(q_1, q) dS_q = -\mathbf{E} \quad (q_1 \in S). \end{aligned} \tag{1.19}$$

Here, $\mathbf{E}$ is a unitary matrix, and $\mathbf{0}$ is the zero matrix.

The Gaussian theorem can be used for finding the properties of the generalized double layer potential. We transform the potential (1.8) as follows:

$$\begin{aligned} \mathbf{W}(p) &= \int_S \mathbf{\Gamma}_2(p, q)\varphi(q) dS_q\\ &= \int_S \mathbf{\Gamma}_2(p, q)[\varphi(q) - \varphi(q_1)] dS_q + \int_S \mathbf{\Gamma}_2(p, q)\varphi(q) dS_q, \end{aligned} \tag{1.20}$$

where q_1 is a certain point on the surface S, in whose neighbourhood we are studying the limiting properties now.

Suppose that the point p approaches the point q_1 along any path (from inside or outside). The first integral in (1.20) is an improper, uniformly converging integral and is therefore a continuous function (of course, under the condition that $\varphi(q)$ belongs to the class H-L). The behaviour of the second integral has already been analyzed.

On the basis of the above, we arrive at the following equations:

$$\mathbf{W}^{+}(q_1) - \mathbf{W}^{-}(q_1) = -2\varphi(q_1), \quad \mathbf{W}^{+}(q_1) + \mathbf{W}^{-}(q_1) = 2\mathbf{W}^{-}(q_1), \tag{1.21}$$

where $\mathbf{W}(q_1)$ is the proper value of the double layer potential. These equations can also be written in a somewhat different form:

$$\mathbf{W}^{+}(q_1) = -\varphi(q_1) + \mathbf{W}(q_1), \quad \mathbf{W}^{-}(q_1) = \varphi(q_1) + \mathbf{W}(q_1). \tag{1.21'}$$

Let us now go over to an investigation of the limiting values of the stress operator for a single layer potential. The integral (1.3) can be represented in the form

$$T_{\nu(p)}\mathbf{V}(p) = \int_S [\mathbf{\Gamma}_1(p, q)\varphi(q) + \mathbf{\Gamma}_2(p, q)\varphi(q_1)]dS_q - \int_S \mathbf{\Gamma}_2(p, q)\varphi(q_1)dS_q, \tag{1.22}$$

where q_1 is a certain point on the surface S, in whose vicinity we shall be investigating the properties in which we are interested. The behaviour of the second integral is determined by the formulas (1.21). The first integral, on the other hand, consists of two groups of terms. The terms appearing in one group have a singularity of the integrable order, but they appear in such a combination that the integral determined by them is found to be uniformly convergent and hence continuous at the points on the surface. The other group consists of terms having a second-order singularity. The integral defined by them is found to be continuous in view of the fact that the conditions for the existence of the singular integral (3.20), Ch. 1, Vol. 1 are satisfied.

The final results are given below:

$$\begin{aligned} [T_\nu \mathbf{V}(q_1)]^{+} - [T_\nu \mathbf{V}(q_1)]^{-} &= 2\varphi(q_1), \\ [T_\nu \mathbf{V}(q_1)]^{+} + [T_\nu \mathbf{V}(q_1)]^{-} &= 2T_\nu \mathbf{V}(q_1). \end{aligned} \tag{1.23}$$

These results can be written in the following alternative form as well:

$$\begin{aligned} [T_\nu \mathbf{V}(q_1)]^{+} &= \varphi(q_1) + T_\nu \mathbf{V}(q_1), \\ [T_\nu \mathbf{V}(q_1)]^{-} &= -\varphi(q_1) + T_\nu \mathbf{V}(q_1). \end{aligned} \tag{1.23'}$$

Here, $T_\nu \mathbf{V}$ is the proper value of the stress operator for a single layer potential.

Naturally, the formulas derived above show that the corresponding limiting values exist.

We shall now formulate the Lyapunov-Tauber theorem without proving it: if the limiting values of the stress operator for a double layer potential exist on one side of the surface, they exist on the other side as well, and the following equality holds:

$$[T_\nu \mathbf{W}]^{+} = [T_\nu \mathbf{W}]^{-}. \tag{1.24}$$

Let us now consider the volume integral

$$\mathbf{U}(p) = \int_\Omega \mathbf{\Gamma}(p, p_1)\varphi(p_1)\, d\Omega_{p_1}, \tag{1.25}$$

called the generalized elastic volume (Newtonian) potential. We shall assume that

the function $\varphi(p)$ is a bounded measurable function, satisfying the condition $|\varphi(p)| \leqslant c/R^3$ at infinity. It is always convenient to assume that the integral is extended over the entire space, and to put the function $\varphi(p)$ equal to zero in the additional domain.

We shall show that for certain conditions on the smoothness of the density this potential satisfies Lamé's inhomogeneous equation whose right-hand side is equal to twice the density [74].

We introduce the operator

$$\Delta^{**} = \frac{1}{a^2}\,\text{grad div} - \frac{1}{b^2}\,\text{curl curl}, \tag{1.26}$$

where $a^2 = \lambda + 2\mu$ and $b^2 = \mu$. It should be recalled that the Lamé operator may be represented in the form (see Sec. 4, Ch. 2, Vol. 1)

$$\Delta^* = a^2\,\text{grad div} - b^2\,\text{curl curl}.$$

Next, we introduce the matrix $\mathbf{\Omega}$ with the elements

$$\Omega_{ij} = \frac{1}{4\pi}\delta_{ij}r. \tag{1.27}$$

By direct verification, we can prove that the following equality is valid:

$$\mathbf{\Gamma}(p,q) = \Delta^{**}\mathbf{\Omega}(p,q). \tag{1.28}$$

By way of an example, let us consider the element Γ_{12}. In accordance with (1.28), we get

$$\begin{aligned}\Gamma_{12} &= \left(\frac{1}{a^2}\,\text{grad}_{x_2}\,\text{div}\mathbf{\Omega}^1 - \frac{1}{b^2}\,\text{curl}_{x_2}\text{curl}\mathbf{\Omega}^1\right)\\ &= \frac{1}{2a^2}\frac{\partial^2 r}{\partial x_1 \partial x_2} - \frac{1}{b^2}\left(\frac{\partial}{\partial x_3}\,\text{curl}_{x_1}\mathbf{\Omega}^1 - \frac{\partial}{\partial x_1}\,\text{curl}_{x_3}\mathbf{\Omega}^1\right)\\ &= \frac{1}{2a^2}\frac{\partial^2 r}{\partial x_1 \partial x_2} - \frac{1}{2b^2}\frac{\partial^2 r}{\partial x_1 \partial x_2} = m\frac{1}{r}\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2},\end{aligned}$$

where $\mathbf{\Omega}^1$ is the vector with components Ω_{11}, Ω_{12}, and Ω_{13}. The expression for the element Γ_{11} can be written in a similar way:

$$\begin{aligned}\Gamma_{11} &= \frac{1}{2a^2}\frac{\partial^2 r}{\partial x_1^2} - \frac{1}{b^2}\left(\frac{\partial}{\partial x_2}\,\text{curl}_{x_3}\mathbf{\Omega}^1 - \frac{\partial}{\partial x_3}\,\text{curl}_{x_2}\mathbf{\Omega}^1\right)\\ &= \frac{1}{2a^2}\frac{\partial^2 r}{\partial x_1^2} + \frac{1}{2b^2}\left(\frac{\partial^2 r}{\partial x_2^2} + \frac{\partial^2 r}{\partial x_3^2}\right)\\ &= \frac{1}{2a^2}\frac{\partial^2 r}{\partial x_1^2} + \frac{1}{2b^2}\left(\frac{2}{r} - \frac{\partial^2 r}{\partial x_1^2}\right) = \frac{n}{r} + \frac{m}{r}\left(\frac{\partial r}{\partial x_1}\right)^2.\end{aligned}$$

Let us now transform the potential (1.25) with the help of Eq. (1.28). We have

$$\mathbf{U}(p) = \int_{\Omega} \Delta^{**}\mathbf{\Omega}(p,p_1)\varphi(p_1)d\Omega_{p_1}. \tag{1.29}$$

Acting on (1.29) by the Lamé operator, we get

$$\Delta^*\mathbf{U}(p) = \Delta^*\Delta^{**}\int_{\Omega} \Omega(p,p_1)\varphi(p_1)d\Omega_{p_1}, \tag{1.30}$$

It can be easily verified that the following equality holds:

$$\Delta^*\Delta^{**} = \frac{1}{2\pi}\begin{pmatrix} \Delta\Delta & 0 & 0 \\ 0 & \Delta\Delta & 0 \\ 0 & 0 & \Delta\Delta \end{pmatrix}. \tag{1.31}$$

With the help of (1.31), we can represent the integral on the right-hand side of (1.30) in the following form:

$$\Delta^*\mathbf{U}(p) = \frac{1}{4\pi}\int_{\Omega} \Delta\Delta r(p,p_1)\varphi(p_1)d\Omega_{p_1} = \frac{\Delta}{2\pi}\int_{\Omega}\frac{\varphi(p_1)}{r(p,p_1)}\,d\Omega_{p_1}. \tag{1.32}$$

In view of Eq. (6.32), Ch. 1, Vol. 1, the last integral is equal to twice the density $\varphi(p)$ with a minus sign.

The apparatus described above can be fully extended to the problems of the theory of vibrations. Naturally, the starting point here is the solution for a periodically changing concentrated force. We shall denote the generalization of the Kelvin-Somilyana matrix by $\boldsymbol{\Gamma}(p, q, \omega)$. The elements of this matrix have the form

$$\Gamma_{ij}(p,q,\omega) = \sum_{l=1}^{2}\left(\delta_{ij}\alpha_l + \beta_l\frac{\partial^2}{\partial x_i\partial x_j}\right)\frac{e^{ik_l r(p,q)}}{r(p,q)}, \tag{1.33}$$

where

$$k_1^2 = \frac{\rho\omega^2}{\lambda + 2\mu}, \qquad k_2^2 = \frac{\rho\omega^2}{\mu},$$

$$\alpha_l = \frac{\delta_{2l}}{2\pi\mu}, \qquad \beta_l = \frac{(-1)^l}{2\pi\rho\omega^2}.$$

The single layer potential, the double layer potential, and the volume potential can be constructed with the help of this matrix. These potentials have qualitatively the same characteristics as the potentials considered above in the problems of statics. The conditions at infinity are identical to the radiation conditions (see Sec. 1, Ch. 3, Vol. 1).

Let us now consider the general dynamic problems. In order to solve these problems, we can either construct the generalized elastic potentials [75] (of the type of delayed potentials of the wave equation, see Sec. 9, Ch. 1, Vol. 1), or, using the representation of displacements in terms of four wave functions (see Sec. 5, Ch. 3, Vol. 1), directly proceed from the wave potentials themselves [76].

A fundamentally different approach has been adopted in [77]. Here, the Laplace transformation in time is carried out and all the constructions are accomplished with the help of displacement transforms. The differential equation obtained for them may be treated as an equation for amplitudes (see Sec. 4, Ch. 2, Vol. 1) with a complex frequency. Hence, it is possible to obtain the solution for the transform in terms of potentials based on the fundamental solution (1.33).

Let us consider a number of results [77] pertaining to the differential properties of elastic potentials. Suppose that $S \in \Pi_{k+1}(\alpha)$, and the density $\varphi \in C^{l,\beta}$ $(0 < \beta < \alpha \leqslant \leqslant 1,\ 0 \leqslant l \leqslant k + 1)$. Then, the double layer potential in a closed domain $\bar{D} = D \cup S$ is given by

$$\mathbf{W}(\varphi) \in C^{l,\beta}. \tag{1.34}$$

If, however, $S \in \Pi_{k+1}(\alpha)$, $\varphi \in C^{l,\beta}$ $(0 < \beta < \alpha \leqslant 1,\ 0 \leqslant l \leqslant k)$, the single layer potential is given by

$$\mathbf{V}(\varphi) \in C^{l+1,\beta}. \tag{1.35}$$

Section 2
Integral Equations for Basic Three-dimensional Problems

The potentials introduced above can be used to reduce the solution of the basic boundary value problems in the theory of elasticity to integral equations of the second kind. We begin with the first basic problem. Suppose that for an elastic body, occupying the domain D bounded by the surface S, it is required to determine the displacements whose limiting values are equal to the given values $\mathbf{F}_1(q)$ (see Eq. (1.1), Ch. 3, Vol. 1). We shall seek the displacements in the form of a generalized double layer elastic potential (1.8). Then, in accordance with the formula (1.21), we arrive at the integral equations

$$\varphi(q) \pm \int_S \mathbf{\Gamma}_2(q, q_1)\varphi(q_1)dS_{q_1} = \mp \mathbf{F}_1(q). \tag{2.1}$$

The upper sign corresponds to the external problem (I^-), while the lower sign corresponds to the internal problem (I^+). By introducing the parameter λ, which assumes the value 1 or -1, we can represent these equations in a unified form corresponding to the traditional form of notation for the Fredholm equations of the second kind:

$$\varphi(q) - \lambda \int_S \mathbf{\Gamma}_2(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}(q). \tag{2.2}$$

The values $\mathbf{F} = -\mathbf{F}_1$ and $\mathbf{F} = \mathbf{F}_1$ correspond to the problems I^+ and I^- respectively.

In order to solve the basic problem II, we should represent the displacements in the form of a generalized single layer potential (1.2). Then, we get integral equations of the second kind from the formulas (1.23). In the universal form, these equations

can be written as follows:

$$\varphi(q) - \lambda \int_S \Gamma_1(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}(q). \tag{2.3}$$

We shall now obtain different singular equations for the problem II.[4] For the internal problem, we turn to the identity (1.12) or (1.13) and carry out the limit transition to the surface points. We assume that the integral of the boundary values of the stresses is known, and represent it by $\Phi(q)$. With the help of the formulas (1.21), we arrive at the following equalities:

$$\mathbf{u}(q) + \int_S \Gamma_2(q, q_1)\mathbf{u}(q_1)dS_{q_1} = \Phi(q), \tag{2.4}$$

which turn out to be singular integral equations in displacements on the boundary surface. The external problem is considered in a similar way (in this case, we use the identity (1.14) or (1.15)). The final result can be written in a unified form as follows:

$$\mathbf{u}(q) - \lambda \int_S \Gamma_2(q, q_1)\mathbf{u}(q_1)dS_{q_1} = \Phi(q). \tag{2.5}$$

The value $\lambda = 1$ corresponds to the problem II^-, while for the problem II^+, $\lambda = -1$.

Equations (2.5) are equivalent to the equations of the problem I, obtained by representing the displacements in the form of a double layer potential. However, these equations differ from the equations for the problem I in the physical meaning of the required function as well as in the values of the right-hand sides.

It should be noted that mathematically, neither of the equations (2.3) and (2.5) has any advantage over the other. Some specific points in this connection will be mentioned in Sec. 3 while comparing these from the point of view of numerical calculations.

It should be observed that Eqs. (2.2) and (2.3) (and similarly Eqs. (2.5) and (2.3)) are companion to each other. The index of a singular equation (the difference in the number of eigenfunctions for the original equation and its companion equation) may be, generally speaking, any integer. We shall show that the index of the singular equations constructed above is equal to zero. According to [78], there exists an operator which transforms them into the equivalent reqular equations of the second kind. Hence Fredholm's alternatives are applicable to the original equations.

As mentioned earlier (see Sec. 3, Ch. 1, Vol. 1), the necessary and sufficient condition that the index be equal to zero for a system of singular equations is that the symbolic determinant must not be equal to zero when the symbolic matrix itself is Hermitian. It should be noted that upon a transformation of variables, the argument of each element of the symbolic determinant undergoes a linear transformation. The argument of the determinant also undergoes such a transformation, and hence the set of its values is invariant to a change of variables. This circumstance permits us to adopt the following method of investigation.

[4] The equations for the problem I can be obtained in a similar way. However, the actual construction of their right-hand sides is quite complicated, and hence we shall not be considering these.

At a certain point q of the surface, we introduce local coordinates by taking the x_1- and the x_2-axes in the tangential plane, and the x_3-axis along the normal. We shall also define the vector $\varphi(q)$ by its projections onto these axes. This does not lead to any variation of the results. The system of equations (2.3) may then be written in the form

$$\varphi_1(q) \pm A \int\limits_S \frac{x_1 - y_1}{r^3} \varphi_3(q_1) dS_{q_1} + T_1\varphi = F_1(q),$$

$$\varphi_2(q) \pm A \int\limits_S \frac{x_2 - y_2}{r^3} \varphi_3(q_1) dS_{q_1} + T_2\varphi = F_2(q),$$

$$\varphi_3(q) \pm A \int\limits_S \frac{(x_1 - y_1)\varphi_1(q_1) + (x_2 - y_2)\varphi_2(q_1)}{r^3} dS_{q_1} + T_3\varphi = F_3(q),$$

where T_i are specified regular operators, and $A = (2\sigma - 1)/(1 - \sigma)$.

The characteristics appearing in this system of singular integrals can be easily determined: $\dfrac{x_1 - y_1}{r} = \cos\theta$, and $\dfrac{x_2 - y_2}{r} = \sin\theta$. The symbolic determinant then acquires the form

$$\begin{vmatrix} 1 & 0 & iA\cos\theta \\ 0 & 1 & iA\sin\theta \\ -iA\cos\theta & -iA\sin\theta & 1 \end{vmatrix} = 1 - A^2 = \frac{3 - 4\sigma}{4(1 - \sigma)^2}.$$

Consequently, this determinant is non-zero for the usual values of Poisson's ratio $(0 \leqslant \sigma < 0.5)$.

Let us now consider the spectral properties of Eqs. (2.2) and (2.3), as well as Eq. (2.5). We put $\lambda = 1$ and assume that these equations have non-trivial eigenfunctions (since the index is equal to zero, the number of these functions is the same). We denote by φ_0 the eigenfunction of Eq. (2.3), and by $\mathbf{V}(p, \varphi_0)$ the potential whose density is given by the function φ_0. This potential is the solution of the problem II^- when the stresses at the surface are equal to zero. It is appropriate to mention here that in any case, the displacements which can be represented as a single layer potential decrease as $1/R$ at infinity, while the stresses decrease as $1/R^2$. Hence we can use the theorem of uniqueness of the external problem in the theory of elasticity. We then find that the potential $\mathbf{V}$ is identically equal to zero in the domain D^-. On the other hand, the single layer potential is a function which is continuous everywhere including the surface S. Hence the potential $\mathbf{V}(p, \varphi_0)$ will be identically equal to zero in the domain D^+, since it vanishes at the surface S. Returning to the formula (1.24), we find that the function φ_0 is identically equal to zero. Consequently, the point $\lambda = 1$ is not an eigenvalue for Eqs. (2.3), (2.4), and (2.5).

Hence these equations are solvable for any right-hand side, and the solutions obtained are unique.

Let us now consider these equations for $\lambda = -1$. It was established in Sec. 1, Ch. 3, Vol. 1 that a solution of the problem II^+ exists for zero stress at the boundary, and has the form

$$u_1 = a_1 + qx_3 - rx_2,\ u_2 = a_2 + rx_1 - px_3, \qquad u_3 = a_3 + px_2 - qx_1, \tag{2.6}$$

where a_1, a_2, a_3, p, q and r are arbitrary constants characterizing the displacement of the body as a rigid entity.

Let $\mathbf{u}_0$ be any displacement of the body as a rigid object. We shall show that the function $\mathbf{u}_0(q)$ is then an eigenfunction of Eq. (2.2). Indeed, from (1.12) we find that

$$\mathbf{u}_0(p) = -\frac{1}{2}\int_S \mathbf{\Gamma}_2(p, q)\mathbf{u}_0(q)dS_q. \tag{2.7}$$

Carrying out the limiting transition to the points on the surface S, we arrive at the identity

$$\mathbf{u}_0(q_1) + \int_S \mathbf{\Gamma}_2(q_1, q)\mathbf{u}_0(q)dS_q = 0, \tag{2.8}$$

expressing the fact that the function $\varphi_0(q)$ is the eigenfunction of Eq. (2.2) for $\lambda = -1$.

We denote by φ_{1i} $(i = 1, 2, \dots, 6)$ the eigenfunctions of Eq. (2.2) for $\lambda = -1$. They can be determined, for example, by putting successively one of the constants $a_1, a_2, \dots, r$ equal to unity and the others equal to zero. We also form the potentials $\mathbf{W}(p, \varphi_{1i})$. It follows from (2.8) that their limiting values from within the surface vanish. It then follows from the uniqueness theorem for the problem I^+ that they are identically equal to zero in the domain D^+.

Since Eq. (2.2) has six eigenfunctions for $\lambda = -1$, its adjoint equation (2.3) will also have six eigenfunctions which we denote by φ_{2i}. Let us form the potentials $\mathbf{V}(p, \varphi_{2i})$. Since these potentials correspond to zero boundary value conditions in stresses, they represent the displacement of the body as a rigid object.[5]

Thus, it has been shown that Eqs. (2.2), (2.3), and (2.5) have eigenfunctions for $\lambda = -1$.

Let us now go over to the derivation of the solvability conditions. In accordance with Fredholm's alternative (see Sec. 2, Ch. 1, Vol. 1), the necessary and sufficient condition for the solvability of equations lying on the spectrum is that the right-hand sides and the eigenfunctions of the adjoint equations be orthogonal. Since the eigenfunctions of Eq. (2.2) have been obtained above, we at once get the conditions of solvability of Eq. (2.3) in the form

[5] In this case, as well as in the subsequent cases, it is appropriate to speak of the displacements of the surface, since the concept of domain must vanish as we go over to integral equations.

$$\int_S \mathbf{F}(q)\varphi_{1i}(q)dS = 0 \qquad (i = 1, 2, \ldots, 6). \tag{2.9}$$

These equations have a quite clear mechanical sense. Since the functions φ_{1i} are linear functions, they can be chosen in the simplest form in accordance with the above description. We then get the equalities

$$\int_S F_i(q)dS = 0, \quad \int_S (F_i x_{i+1} - F_{i+1}x_i)dS = 0$$
$$\int_S (F_i x_{i+2} - F_{i+2}x_i)dS = 0 \qquad (i = 1, 2, 3), \tag{2.10}$$

where F_i are the components of the vector $\mathbf{F}(q)$. For the sake of simplicity of notation, we have put $x_4 = x_1, x_5 = x_2, F_4 = F_1$, and $F_5 = F_2$ in (2.10). It is now clear that these conditions represent the equality to zero of the principal vector and the vector moment of the forces applied to the body, as required by the statement of the problem (see Sec. 1, Ch. 3, Vol. 1). Hence it is always necessary to require that these conditions be satisfied. Consequently, Eq. (2.3) is always solvable for $\lambda = -1$.[6] In this case the solution is found to be no longer unique. To any particular solution, we must add all its eigenfunctions. But as was shown above, the potentials $\mathbf{V}(p, \varphi_{1i})$ correspond to the displacement of the body as a rigid object, and hence they do not affect the state of stress. In view of this, there is no need to construct the eigenfunctions.

Let us now go over to the problem I^-. In this case, the representation of displacements as a double layer potential at once leads to a restriction on the behaviour of the solution at infinity ($|\mathbf{u}(p)| < c/R^2$), although no such restriction is required in the formulation of the problem. Hence Eq. (2.2) may not be solvable. It should be noted that the very proof of this fact is a quite complicated problem, since it is necessary to determine the eigenfunctions of the companion equation.

There are a number of approaches, consisting of modifications of representations for displacements (or of other methods), which always lead to solvable equations. For example, we can choose any point p_0 in the domain D^+, specify the concentrated force and moment at this point and superimpose the displacements thus obtained onto the required ones. As a result, we arrive at the same integral equation (2.2), but the boundary condition will now contain six numerical parameters which determine the force and the moment. It is possible to find these parameters from the condition that the orthogonality relations are satisfied.

Let us consider one more approach [79] in which the displacements are sought in the form of the sum of double- and single layer potentials (having the same density which is to be determined). In this case, we always get a solvable equation.[7]

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[6] The existence of a resolvent pole at the point $\lambda = -1$, however, introduces certain difficulties in view of the error introduced by numerical calculations. This question will be discussed in detail in Sec. 3.

[7] The problem I^- will be considered in detail at a later stage as a special case of the problem in which the domain is bounded by several surfaces.

Let us now return to the problem II$^+$, but in this case we consider Eq. (2.5). The right-hand side of this equation has a quite complicated structure (on account of this, the conditions of self-balancing of external forces are not imposed explicitly). Besides, the eigenfunctions of the companion equation are also not known. The conditions of solvability of Eq. (2.5) can be written in the form

$$\int_S \varphi_{2i}(q)\Phi(q)dS_q = \int_S \varphi_{2i}(q)\int_S \Gamma(q, q_1)\mathbf{F}_2(q_1)dS_{q_1}dS_q = 0. \qquad (2.11)$$

Changing the order of integration in (2.11), we get

$$\int_S \varphi_{2i}(q)\Phi(q)dS_q = \int_S \mathbf{F}_2(q)\mathbf{V}(q, \varphi_{2i})dS_q. \qquad (2.12)$$

Since, as mentioned earlier, the potentials $\mathbf{V}(q, \varphi_{2i})$ are the displacements of the body as a rigid object, we find that the conditions (2.11) are identical to the conditions (2.9).

The properties of integral equations described above provide an exhaustive answer to the question of their solvability (with the exception of the case of the I$^-$ problem). While actually constructing the solution, it is desirable to have a proof of the method of successive approximations,[8] for which purpose a complete investigation of the spectral properties of these equations (in the plane of the complex variable λ) is essential.

With the help of formula (1.24), we can rewrite Eq. (2.3) in the following form:

$$(1 - \lambda)[T_\nu \mathbf{V}]^+ - (1 + \lambda)[T_\nu \mathbf{V}]^- = 2\mathbf{F}(q). \qquad (2.13)$$

We assume that the integral equations under consideration have complex eigenvalues. We denote one of these by $\lambda_0 = a + ib$, and let $\varphi_0(q) = \varphi_a + i\varphi_b$ be any eigenfunction. We form the single layer potential $\mathbf{V}_0 = \mathbf{V}(p, \varphi_0)$ and substitute it into (2.13). This gives

$$(1 - \lambda_0)\{[T_\nu \mathbf{V}_a]^+ + i[T_\nu \mathbf{V}_b]^+\} = (1 + \lambda_0)\{[T_\nu \mathbf{V}_a]^- + i[T_\nu \mathbf{V}_b]^-\}. \qquad (2.13')$$

Multiplying this equation by $\mathbf{V}_a - i\mathbf{V}_b$ and integrating the resulting equation over the surface S, we get

$$\begin{aligned}(1 - \lambda_0)\int_S &\{\mathbf{V}_a[T_\nu \mathbf{V}_a]^+ + \mathbf{V}_b[T_\nu \mathbf{V}_b]^+\}dS \\ &+ i(1 - \lambda_0)\int_S \{\mathbf{V}_a[T_\nu \mathbf{V}_b]^+ - \mathbf{V}_b[T_\nu \mathbf{V}_a]^+\}dS \\ &= (1 + \lambda_0)\int_S \{\mathbf{V}_a[T_\nu \mathbf{V}_a]^- + \mathbf{V}_b[T_\nu \mathbf{V}_b]^-\}dS \\ &+ i(1 + \lambda_0)\int_S \{\mathbf{V}_a[T_\nu \mathbf{V}_b]^- - \mathbf{V}_b[T_\nu \mathbf{V}_a]^-\}dS. \qquad (2.14)\end{aligned}$$

We can transform this identity by using Betti's second formula (4.26′), Ch. 2, Vol. 1, which is applied to the displacements $\mathbf{V}_a$ and $\mathbf{V}_b$. This gives the equalities

[8] This question will be considered in detail in Sec. 3.

$$\begin{aligned}
&\int_S \{\mathbf{V}_a[T_\nu \mathbf{V}_b]^+ - \mathbf{V}_b[T_\nu \mathbf{V}_a]^+\}dS = 0, \\
&\int_S \{\mathbf{V}_a[T_\nu \mathbf{V}_b]^- - \mathbf{V}_b[T_\nu \mathbf{V}_a]^-\}dS = 0,
\end{aligned} \tag{2.15}$$

with the help of which we can rewrite Eq. (2.14) in the form

$$(1 - \lambda_0) \int_S \{\mathbf{V}_a[T_\nu \mathbf{V}_a]^+ + \mathbf{V}_b[T_\nu \mathbf{V}_b]^+\}dS$$
$$= (1 + \lambda_0) \int_S \{\mathbf{V}_a[T_\nu \mathbf{V}_a]^- + \mathbf{V}_b[T_\nu \mathbf{V}_b]^-\}dS. \tag{2.16}$$

Consequently, the ratio $(1 - \lambda_0)/(1 + \lambda_0)$ is a real number, and hence λ_0 is a real number too. Thus, it is proved that the integral equations (2.8) and (2.9) do not have any complex eigenvalues.

Equation (2.16) can be used to obtain one more result. For this purpose, we make use of Betti's third formula (4.26′), Ch. 2, Vol. 1, and apply it to the displacements $\mathbf{V}_a$ and $\mathbf{V}_b$:

$$\left.\begin{aligned}
&\int_S \mathbf{V}_c[T_\nu \mathbf{V}_c]^+ dS = \int_{D^+} W(\mathbf{V}_c) d\Omega \\
&\int_S \mathbf{V}_c[T_\nu \mathbf{V}_c]^- dS = -\int_{D^-} W(\mathbf{V}_c) d\Omega
\end{aligned}\right\} \quad (c = a, b). \tag{2.17}$$

Thus, the right-hand side of (2.16) is a positive quantity, while the left-hand side is negative. Hence the ratio $(1 - \lambda_0)/(1 + \lambda_0)$ is a negative number. Consequently, $|\lambda_0| \geqslant 1$. On the other hand, the point $\lambda = 1$ corresponds to the problems I^+ and II^-, whose solutions are unique. If we assumed that this point is a resolvent pole, we would find that the boundary value problem is not unique. The situation is different for the point $\lambda = -1$, which corresponds to the problems I^- and II^+. If this point were not a resolvent pole, the integral equation of the problem II^+ would be solvable for any right-hand side, and the boundary value problem would also be always solvable, which is in contradiction with the existence theorem. Consequently, the point $\lambda = -1$ must be a resolvent pole. Since the equations for the problem II^+ are companion equations, and the Fredholm alternatives are satisfied, the integral equation in this case is also solvable only under certain boundary conditions, which are not necessary conditions for the initial boundary value problem.[9]

It has been proved [80] that the pole at the point $\lambda = -1$ is simple. It has been shown above that six eigenfunctions correspond to this point.

It can be seen that the results described above establish an almost complete analogy between the properties of the integral equations for the Dirichlet and

[9] This discrepancy between the solvability of an integral equation and the solvability of the boundary value problem should be treated as a "defect" of the representation for the displacements used here in the form of a double layer potential. This potential decreases at infinity as $1/R^2$, while in accordance with the formulation of the problem, the displacements are of the order $1/R$.

Neumann problems on the one hand, and the basic problems in the theory of elasticity on the other hand.

Let us consider the algorithm for solving Eqs. (2.2), (2.3), and (2.5) by the method of successive approximations [81, 82]. For this purpose, we represent the solution in the form of the series

$$\varphi(q) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(q). \tag{2.18}$$

Substituting this series into the equations and equating the coefficients of like powers of λ, we arrive at the recurrence relations

$$\varphi_n(q) = \int_S \Gamma_j(q, q_1)\varphi_{n-1}(q)dS_q \tag{2.19}$$

$$(n = 1, 2, \dots, j = 1, 2; \varphi_0(q) = \mathbf{F}(q)).$$

The spectral properties of the resolvent of the equations under consideration lead to the following statement (see Sec. 2, Ch. 1, Vol. 1): the solution of the integral equation in the case of the problems I^+ and II^- may be represented in the form of the series[10]

$$\varphi(q) = 0.5\varphi_0 + 0.5(\varphi_0 + \varphi_1) + 0.5(\varphi_1 + \varphi_2) + \dots . \tag{2.20}$$

In the case of the problem II^+, the solution can be represented in the form of the series

$$\varphi(q) = \varphi_0 - \varphi_1 + \varphi_2 - \varphi_3 + \dots . \tag{2.21}$$

As a matter of fact, although the point $\lambda = -1$ is also the resolvent pole, the conditions (2.9) lead to its virtual elimination. Of course, the error introduced during numerical computations may result in a violation of the orthogonality conditions, thus leading to a divergence of the series. This question will be considered in greater detail in Sec. 3.

It was mentioned earlier (see Sec. 2, Ch. 1, Vol. 1) that the process of successive approximations in the case of non-orthogonality of the right-hand side to the eigenfunctions of companion equation (for equations in which the modulo lowest eigenvalue is 1 or -1) leads to the construction of one of the eigenfunctions at the point 1 or -1. This situation has a special interest and will be used later for constructing the solution of the boundary value problems and for estimating the effect of the errors in calculation schemes on the convergence of algorithms.

Suppose that the problem II^+ has been formulated for the case when the boundary conditions are not self-balanced (at least one of the conditions (2.1) is not satisfied). Naturally, the boundary value problem does not have a solution. Let us, however, carry out the process of successive approximations. This leads to a certain eigenfunction $\varphi^*(q)$. The sum $\varphi^N(q) = \sum_{n=0}^{N} (-1)^n \varphi_n(q)$ must diverge with increas-

[10] Other representations are also possible [12, 83].

ing N. The following representation is obviously valid:

$$\varphi^N(q) = \Phi(q) + N\varphi^*(q) + \varepsilon^N(q), \tag{2.22}$$

where $\Phi(q)$ is a certain function, and the function $\varepsilon^N(q) \to 0$ as $N \to \infty$.

Next, let us go over to a single layer potential with density $\varphi^N(q)$ (for sufficiently large values of N). Since the function $\varphi^*(q)$ is an eigenfunction, the single layer potential corresponding to it leads to zero stresses. Hence the process of successive approximations leads, in the case of non-self-balanced boundary conditions, to a solution (in stresses) of a certain boundary value problem whose boundary conditions are defined by the function $\Phi(q)$ and are naturally different from the initial conditions.

It is well known that mixed problems (see Sec. 5) are sometimes solved by series expansion of the boundary conditions in terms of stresses (in that part of the boundary where displacements are given). This is followed by a solution of the set of second boundary value problems. It follows from the above that there is no need to ensure that each harmonic must lead to self-balanced conditions on the whole.

The above considerations can be also used for solving the problem I$^-$ on the basis of Eq. (2.4). In this case we must construct all the six eigenfunctions and hence five more boundary conditions must be specified in such a way that all the six conditions are linearly independent. As a result of iterative processes, we get six eigenfunctions. After this, we must superimpose the six solutions of the boundary value problems (whose solutions are known) with unknown coefficients on the solution of the original problem. These coefficients can be determined from the system of six equations obtained from the orthogonality condition of the modified right-hand side to the six eigenfunctions which are now known.

Let us now consider the problems in the theory of elasticity when the medium is nearly incompressible (σ is nearly equal to 0.5). It should be noted that for an incomporessible medium, the integral equations (2.2) and (2.3) are regular and identical to the equations corresponding to the linearized flow of a viscous incompressible liquid [84]. It has been shown in [84] that at the point -1, these equations have a simple resolvent pole having six eigenfunctions corresponding to it. Besides, these equations also have a simple pole at the point 1 with one eigenfunction corresponding to it. This function is known for the second boundary value problem. It is a vector function directed along the normal to the surface and having a constant modulus. This statement follows from the fact that for a conjugate equation corresponding to the first boundary value problem, the internal problem can be formulated under the condition

$$\int_S \mathbf{F}_1(q)\nu(q)dS_q = 0,$$

expressing the invariance of volume. Hence, the conjugate equation (for problem II$^-$) may be constructed only when certain restrictions are imposed on the boundary condition (although a solution of the boundary value problem exists for the general case also).

As already mentioned, all resolvent poles of singular equations in the theory of elasticity, with the exception of one, are modulo greater than unity. It can be con-

cluded from the above that as σ approaches the value 0.5, the second highest resolvent pole tends to unity. Hence it can be expected that for values of σ close to 0.5, the convergence of series (2.20) and (2.21) will not be satisfactory.

The above discussion necessitates the introduction of certain corrections to the solutions by the method of successive approximations only in the case of the problem II^-. In the case of the I^+ problem, there is no need for any variations, while the following series should be used for the II^+ problem:

$$\varphi = 0.5\varphi_0 + 0.5(\varphi_0 - \varphi_1) - 0.5(\varphi_1 - \varphi_2) + \dots . \tag{2.23}$$

As mentioned above, a realization of the recurrence relations in the II^- problem will lead to the construction of the eigenfunction $C\nu(q)$ or, more precisely, to the determination of the constant C. We shall use this representation for obtaining a convergent representation for the solution [85]. Let us now consider the boundary value problem for the case when the exact solution $\mathbf{u}_1$ in displacements and stresses is known[11]. By realizing the recurrence relations (2.19), we arrive at the corresponding value of the constant (which we denote by C_1). It can be easily seen that the boundary value problem for the displacement $\mathbf{u}_2 = \mathbf{u} - C\mathbf{u}_1/C_1$ will lead to a converging process.

If the convergence of the series (2.20), (2.21), or of their modifications, is not sufficient, it is appropriate to increase the convergence by using the well-known methods (see Sec. 2, Ch. 1, Vol. 1).

Let us now go over to a consideration of the problems when the domain is bounded by several surfaces S_j ($j = 0, 1, \dots, m$), all of which are situated one within the other with the exception of S_0 within which all the other surfaces are contained. It should be noted that the surface S_0 may be missing. For the sake of definiteness, we shall consider the second basic problem. We specify on each surface S_j a vector function $\varphi_j(q)$ which is yet to be determined, and form a single layer potential by treating these functions as densities. Then, carrying out the limit transition to the surface points for the stress operator, we arrive at a system of integral equations, which can be symbolically written in the form

$$\varphi(q) - \lambda \int_S \mathbf{\Gamma}_1(q, q_j)\varphi(q_j)dS_{q_j} = \mathbf{F}(q) \qquad (q, q_j \in S_j). \tag{2.24}$$

Here, by S we mean the combination of S_j's, and $\lambda = -1$. The direction away from the region occupied by the elastic medium is taken as the positive direction of the normal.

It has been mentioned earlier that all Betti's formulas which are applicable for the spectral analysis are also found to be valid for a domain bounded by several surfaces. Hence the statements concerning the real nature of the eigenvalues and of their magnitude (whose modulus is not less than unity) are found to be valid. All

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[11] It is quite simple to proceed as follows: we specify a solution in space, corresponding to the singularity at a certain point in the additional domain (for example, the solution which is written in the form $\sigma_r = -1/r^3$, $\sigma_\theta = \sigma_\varphi = 1/2r^3$, $\tau_{r\varphi} = \tau_{r\theta} = \tau_{\theta\varphi} = 0$ in spherical coordinates), and determine the stresses introduced by it on the surface S.

that is left is to additionally analyze the points $\lambda = \pm 1$.

The homogeneous integral equation, companion to (2.24), is an equation which may be obtained if we try to construct the solution of the first basic problem for the domains $D_1^+, D_2^+, \ldots, D_m^+$ in the form of a generalized double layer elastic potential, distributed over all surfaces.[12] Since the boundary conditions are homogeneous, all the displacements in the additional domains will be equal to zero, and hence the stresses will be also equal to zero. It follows from the continuity of the stress vector at the boundary that the stresses are equal to zero everywhere in the domain D, thus resulting in the displacements of the body as a rigid object. Since a non-trivial solution exists under homogeneous conditions, Eq. (2.24) is solvable in the general case if the orthogonality conditions (2.9) are satisfied, where by S we mean the totality of all surfaces. Apparently, if the surface S_0 exists, a rigid displacement of the domain D_0^- is not permissible, and hence there are no non-trivial solutions. Consequently, the equation is solvable for any boundary conditions.

The integral equation (2.24) for $\lambda = 1$ corresponds to the second basic problem for the set of domains $D_1^+, D_2^+, \ldots, D_m^+$, when the solution is sought in the form of a unique single layer potential, distributed over all the surfaces. The eigenfunctions of the companion equation correspond to the solution of the first basic problem for the domain D. Using the generalized Gauss theorem (1.19), we can show without any difficulty that the displacement of each of the surfaces S_j ($j \neq 0$) as a rigid body is an eigenfunction. Hence, unlike the case when the domain is bounded by one surface, the point $\lambda = 1$ is a resolvent pole.

On the basis of the above, we can state that a solution of Eq. (2.24) in its direct form (2.18) by the method of successive approximations is possible if the load applied to each of the surfaces is self-balanced (this practically results in the vanishing of the resolvent pole at the point $\lambda = 1$). If this is not so (of course, the system of forces should be balanced on the whole if the surface S_0 exists), we must use the representation (2.20).

Let us now consider one special case. If the surface S is a sphere of radius R, Eqs. (2.2) and (2.3) can be solved in an explicit form. It is easiest to obtain a solution when the load is reduced to a hydrostatic pressure p^+. The boundary conditions are then given by

$$T_{\nu_r}\mathbf{u} = p^+, \qquad T_{\nu_\varphi} = T_{\nu_\theta} = 0. \tag{2.25}$$

The solution of this problem for physical quantities is trivial:

$$\begin{gathered} u_r = \frac{1-2\sigma}{E}p^+ r, \qquad u_\varphi = u_\theta = 0, \qquad \sigma_r = p^+, \\ \sigma_\theta = \sigma_\varphi = p^+, \qquad \tau_{r\varphi} = \tau_{r\theta} = \tau_{\theta\varphi} = 0. \end{gathered} \tag{2.26}$$

The displacements indicated above are a single layer potential which assumes the following value on the surface:

$$\frac{1-2\sigma}{E}p^+ R = \alpha R \qquad \left(\alpha = \frac{1-2\sigma}{E}p^+\right) \tag{2.27}$$

[12] There is no need to adopt such a procedure if we simply want to obtain a solution in each domain. The problems can then be formulated and solved independently.

(we mean the only non-zero component u_r). Since this potential is a function which is continuous everywhere in space, the problem of its determination in the external domain consists in the solution of the first basic problem for a space with a spherical cavity, when the displacement (2.27) is given on the surface. The solution of this problem can be obtained in the following way. Suppose that a certain hydrostatic pressure p^- is applied to a surface of the cavity of radius R. The solution then has the form

$$u_r^- = -\frac{1+\sigma}{2E}p^-\frac{R^2}{r}, \quad u_\theta^- = u_\varphi^- = 0. \tag{2.28}$$

Equating the displacements $u_r^-|_{r=R} = \alpha R$, we get the relation

$$p^- = -2\frac{1-2\sigma}{1+\sigma}p^+,$$

whose substitution into (2.28) leads to a representation of the potential in the external domain.

Thus, we have obtained the representations for the potential in the internal and the external domain. Moreover, we have determined the limiting values, from inside and outside, for the stress operator $\left(p^+ \text{ and } p^- = \frac{1-2\sigma}{1+\sigma}p^+\right)$. With the help of the dependence (1.23), we can get the value of the density of this potential, i.e. the solution of the corresponding integral equation in the form

$$2\varphi_r = p^- - p^+ = \frac{3(1-\sigma)}{1+\sigma}p^+. \tag{2.29}$$

Here also, we take into consideration the normal component only.

If the above discussion is carried out by starting from the external problem, we get the solution of the integral equation for the external problem. The result can be written in the following form:

$$\varphi_r = \frac{3(1-\sigma)}{4(1-2\sigma)}p^-. \tag{2.30}$$

We shall make a few observations of general nature [77]. We have considered above the solution of the basic boundary value problems in the theory of elasticity on the basis of the representation of displacements in the form of the corresponding potentials. We have obtained singular integral equations and established the conditions for their solvability under the assumption that the boundary surface belongs to the class of Lyapunov surfaces and the right-hand side belongs to the H-L class. In this case, the solution also belongs to the H-L class.

Let us suppose that the problem II is being solved. Then the density of the single layer potential (i.e. the solution of the integral equation) belongs to the class $C^{0,\beta}$ and, according to what has been stated in Sec. 1, the single layer potential will be a function of the class $C^{1,\beta}$ which will be the regular (classical) solution of the boundary value problem. A similar analysis of the problem I will not lead to the same result. Since the density belongs to the class $C^{0,\beta}$ as before, the double layer poten-

tial will belong to the class $C^{0,\beta}$, which is not a regular solution. In this case, we must speak of the solution as a generalized one.

It should be noted that if additional restrictions are imposed on the surface and the boundary conditions, we can arrive at the classical solution. Indeed, suppose that $S \in \mathit{\Pi}_2(\alpha)$ and $\mathbf{F}(q) \in C^{1,\beta}$. Then the solution of the integral equation will belong to the class $C^{1,\beta}$ and the double layer potential will be a regular solution (i.e. a function of the class $C^{1,\beta}$).

It should be noted that regular integral equations were also obtained for the problems I and II on the basis of potentials chosen in a special way (see, for example, [72] and [86]). Naturally, the general result that the boundary value problems can be reduced for the systems of differential equations to Fredholm's integral equations is valid for the problems in the theory of elasticity [87].

Let us now consider the application of Green's matrix (tensor) in the theory of elasticity. This matrix is defined in the following way. Suppose that p is a certain point in the domain D and $\Gamma(p, q)$ is its corresponding Kelvin-Somilyana solution. Suppose that $\mathbf{U}(p, q)$ is a certain matrix each of whose columns satisfies Lamé's equations (in the coordinates of the point q), while the point p appears as a parameter in the elements of this matrix. It can then be shown (by repeating practically all the discussion that led to the formula (1.12)) that the following equality holds:

$$\mathbf{u}(p) = \int\limits_S \mathbf{G}(p, q)\mathbf{T}_\nu \mathbf{u}(q) dS_q - \int\limits_T \mathbf{T}_\nu \mathbf{G}(p, q)\mathbf{u}(q) dS_q, \tag{2.31}$$

where

$$\mathbf{G}(p, q) = \Gamma(p, q) - \mathbf{U}(p, q).$$

If, in addition, we require that $\mathbf{G}(p, q) = 0$ at the surface points, we get

$$\mathbf{U}(p, q) = \Gamma(p, q) \quad (q \in S),$$

and arrive at Green's matrix for the first basic problem of the theory of elasticity. In this case, the representation (2.31) is simplified and assumes the form

$$\mathbf{u}(p) = -\int\limits_S \mathbf{T}_\nu \mathbf{G}(p, q)\mathbf{u}(q) dS_q, \tag{2.31'}$$

which is practically an explicit representation for the solution of the first basic problem. The existence of the matrix $\mathbf{U}(p, q)$ follows from the existence of the solution of the first basic problem. Consequently, Green's matrix also exists.

Green's matrix can be also used to obtain a representation for displacements satisfying Lamé's inhomogeneous equations ($\Delta^* \mathbf{u} = f(q)$):

$$\mathbf{u}(p) = -\int\limits_S \mathbf{T}_\nu \mathbf{G}(p, q)\mathbf{u}(q) dS_q + \int\limits_D \mathbf{G}(p, q)\mathbf{f}(q) d\Omega. \tag{2.32}$$

The construction of Green's matrix for the second basic problem involves certain difficulties (similar to the ones encountered while constructing Green's function for the Neumann problem (see Sec. 7, Ch. 1, Vol. 1). As a matter of fact, when the domain D is finite, we cannot choose the matrix $\mathbf{U}(p, q)$ in such a way that the stress operator of the displacements given by the matrix $\mathbf{G}(p, q)$ should vanish at the surface.

The apparatus of Green's functions can be applied for the analysis of the problems in the theory of vibrations. The equation for the displacement amplitudes,

$$\Delta^*\mathbf{u} = -k^2\mathbf{u}, \tag{2.33}$$

enables us to represent the displacements in terms of Green's matrix (2.32):

$$\mathbf{u}(p) = -\int_S \mathbf{T}_\nu \mathbf{G}(p, q)\mathbf{u}(q)dS_q + k^2\int_D \mathbf{G}(p, q)\mathbf{u}(q)d\Omega. \tag{2.34}$$

Consequently, we get the following symmetric integral equation for homogeneous boundary conditions:

$$\mathbf{u}(p) = k^2\int_D \mathbf{G}(p, q)\mathbf{u}(q)d\Omega. \tag{2.35}$$

It can be shown [88] that the kernel is square integrable. Hence, by applying the theory of Fredholm's symmetric integral equations, we arrive at the proof of the existence (for a finite domain) of a discrete spectrum of eigenvalues (or of natural vibration frequencies), which are real and even positive numbers[13].

A similar situation arises for the second basic problem as well.

In order to investigate the boundary value problems, we now construct singular integral equations on the basis of single- and double layer potentials (starting from the matrix (1.33)). The extension of Fredholm's alternatives to these equations takes place automatically, since the equations themselves differ from the equations of statics only in that regular terms are present. Complications arise on account of the fact that the solutions of the homogeneous problems may not be unique for certain values of the natural vibration frequency k^2.

It is interesting to note the following point, which arises from the correspondence between the integral equations of the problems I^+ and II^-. Suppose that k^2 is the natural vibration frequency. Then the integral equation for the problem I^+ will have a non-trivial solution. But it follows from the solvability of the boundary value problem that the orthogonality conditions will be satisfied in this case. Since the integral equation for the problem II^- is companion to this equation, the number k^2 will also appear in the spectrum, but the orthogonality conditions need not be satisfied in this case, and hence the equation cannot be solved. In this case, we must modify the representation for the amplitudes by introducing certain terms in it [77].

Let us now consider the construction of singular integral equations (one-dimensional, of course) for the plane problem in the theory of elasticity[14]. It should be recalled that regular equations were constructed and investigated in Sec. 3, Ch. 5.

As in the three-dimensional case, we shall construct singular integral equations by using the solution for a concentrated force (in an infinite plane), which will be represented in terms of the matrix $\tilde{\mathbf{\Gamma}}(p, q)$, called Boussinesq's matrix:

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[13] The proof of this statement requires the application of Betti's identity (in this case, we have a complete analogy with the case of Helmholtz's equation; see Sec. 7, Ch. 1, Vol. I).

[14] In [89], the equations have been constructed and analyzed for the case of periodic vibrations.

$$\tilde{\Gamma}(p,q) = \frac{\lambda+\mu}{2\pi\mu(\lambda+2\mu)}\begin{Vmatrix} \dfrac{\lambda+3\mu}{\lambda+\mu}\ln r - \left(\dfrac{\partial r}{\partial x_1}\right)^2 & -\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2} \\ -\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2} & \dfrac{\lambda+3\mu}{\lambda+\mu}\ln r - \left(\dfrac{\partial r}{\partial x_2}\right)^2 \end{Vmatrix}. \tag{2.36}$$

The elements of this matrix will be constructed, in particular, by proceeding from the solution represented in complex form (see Eq. (4.72), Ch. 5).

Next, for considering the second boundary value problem, for example, we construct the single layer potential

$$\mathbf{V}(p) = \int_L \tilde{\Gamma}(p,q)\varphi(q)dL_q, \tag{2.37}$$

where L is the boundary contour.

The action of the stress operator (for the two-dimensional case) on the potential (2.37) and the limit transition to the contour points lead to the following singular integral equation:

$$\varphi(q) - \lambda\int_L \tilde{\Gamma}_1(q,q_1)\varphi(q_1)dL_{q_1} = \mathbf{F}(q). \tag{2.38}$$

For the internal problem, the parameter $\lambda = -1$, while the function $\mathbf{F}(q)$ is identical to the boundary condition. For the external problem, $\lambda = 1$ and the function $\mathbf{F}(q)$ is opposite in sign to the boundary condition.

The matrix elements $\tilde{\Gamma}_1(q,q_1) = T_{\nu(q)}\tilde{\Gamma}(q,q_1)$ are given as follows:

$$\tilde{\Gamma}_1(q,q_1) = \begin{Vmatrix} m_1 + n_1\left(\dfrac{\partial r}{\partial x_1}\right)^2 & n_1\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2} \\ n_1\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2} & m_1 + n_1\left(\dfrac{\partial r}{\partial x_2}\right)^2 \end{Vmatrix}\frac{d}{dn(q)}\ln r + m_1\begin{Vmatrix} 0 & \omega(q,q_1) \\ -\omega(q,q_1) & 0 \end{Vmatrix}\ln r.$$

Here,

$$m_1 = \frac{\mu}{\pi(\lambda+2\mu)}, \quad n_1\frac{2(\lambda+\mu)}{\pi(\lambda+2\mu)},$$

$$\omega(q, q_1) = \frac{\partial}{\partial x_2} \cos(n_q, x_1) - \frac{\partial}{\partial x_1} \cos(n_q, x_2).$$

We shall mention some new features. Here, there is a complete clarity in respect to the Lyapunov-Tauber theory. If the density function together with its derivative satisfies the H-L condition, there exist the limiting values of the stress operator and they are equal (from the two sides). In the case of the exterior problem II$^-$, for the solution to be regular at infinity, the resultant vector of the forces must vanish. The question of the index of the resulting system of one-dimensional singular equations is solved on the basis of the well-known results of the theory of systems of this kind. A direct calculation shows that this index is zero, and hence the Fredholm alternatives apply to the systems of singular integral equations of the plane problem in elasticity, and this in combination with the uniqueness theorems enables one to answer the question of their solvability.

As in the three-dimensional case, the integral equations for the second fundamental problem can be constructed on the basis of Betti's formula (in its two-dimensional form):

$$\mathbf{u}(q') + \int_L \tilde{\Gamma}_2(q', q)\mathbf{U}(q)dS_q = \int_L \tilde{\Gamma}(q', q)\mathbf{F}(q)dS_q,$$

where $\tilde{\Gamma}_2(q', q)$ denotes a matrix companion to $\tilde{\Gamma}_1(q', q)$.

In conclusion, it might be well to point out the research specially devoted to the construction and investigation of (regular and singular) integral equations for axially symmetric problems and also to the development of methods for their solution [90, 91, 92].

Section 3
Methods of Numerical Realization

Numerical methods for solving integral equations are based primarily on the possibility of evaluating the integrals appearing in these equations, irrespective of the method of solving the equations, be it the method of successive approximations (when all the integrands are known at every stage) or the method of mechanical quadratures (when the required function is assumed to be constant within a small domain, thus enabling us to take the function outside the integral sign), we always end up with the computation of the integral of a known expression.

The integrals appearing in Eqs. (2.2), (2.3), and (2.5) are two-dimensional singular integrals. In accordance with the general theory (see Sec. 3, Ch. 1, Vol. 1), in order to calculate these integrals, we should introduce each time a local system of coordinates defined by the intersection of the surface with the coordinate planes (r = const, φ = const) of a cylindrical system of coordinates whose z-axis coincides with the normal to the surface at the point where the integral is computed.

This method involves serious technical difficulties, which become even more significant as we go over to the solution of an integral equation for which the singular integrals have to be computed at a large number of points on the surface. However, these difficulties were overcome by taking into account the properties of the kernels of these integrals. One method [93] consists in the transformation of these singular integrals into improper (regular) integrals, while another [94, 95] is based on the possibility of computing the integral of the kernel in an explicit form, when the surface element is a plane polygon.

Let us first consider the first method. For integrals appearing in Eqs. (2.2), (2.3), and (2.5), the following identities are valid:

$$\int_S \mathbf{\Gamma}_2(q_1, q)\varphi(q)dS_q = -\varphi(q_1) + \int_S \mathbf{\Gamma}_2(q_1, q)[\varphi(q) - \varphi(q_1)]dS_q, \quad (3.1)$$

$$\int_S \mathbf{\Gamma}_1(q_1, q)\varphi(q)dS_q = -\varphi(q_1) + \int_S [\mathbf{\Gamma}_1(q_1, q)\varphi(q) - \mathbf{\Gamma}_2(q_1, q)\varphi(q_1)]dS_q. \quad (3.2)$$

These identities are called regular representations.

If we assume that the functions $\varphi(q)$ belong to the H-L class, the integrals appearing on the right-hand sides are apparently improper and hence some cubature formulas can be used for computing these integrals.

Let us now consider the case when the surface S is not closed[15] [96]. In order to construct regular representation (at the internal points of the surface S) for the singular integrals under consideration, we must somehow extend the surface S to the closed surface S_2 ($S_2 = S \cup S_1$) by putting the density on S_1 equal to zero. We then get the regular representations in the following form:

$$\int_S \mathbf{\Gamma}_2(q_1, q)\varphi(q)dS_q = -\varphi(q_1) + \int_S \mathbf{\Gamma}_2(q_1, q)[\varphi(q) - \varphi(q_1)]dS_q - \int_{S_1} \mathbf{\Gamma}_2(q_1, q)dS_q\varphi(q_1), \quad (3.1')$$

$$\int_S \mathbf{\Gamma}_1(q_1, q)\varphi(q)dS_q = -\varphi(q_1) + \int_S [\mathbf{\Gamma}_1(q_1, q)\varphi(q) - \mathbf{\Gamma}_2(q_1, q)\varphi(q_1)]dS_q - \int_{S_1} \mathbf{\Gamma}_2(q_1, q)dS_q\varphi(q_1). \quad (3.2')$$

[15] This method is important for problems where the configuration of the body and the boundary conditions permit the density $\varphi(q)$ to be put equal to zero at a certain part of the boundary surface. Besides, the computation of integrals on unclosed surfaces is encountered while considering mixed problems (see Sec. 5).

It should be noted that if S is a part of a plane [97], the above formulas are simplified and assume the form

$$\int_S \Gamma_2(q_1, q)\varphi(q)dS_q = \int_S \Gamma_2(q_1, q)[\varphi(q) - \varphi(q_1)]dS_q, \tag{3.1''}$$

$$\int_S \Gamma_1(q_1, q)\varphi(q)dS_q = \int_S [\Gamma_1(q_1, q)\varphi(q) - \Gamma_2(q_1, q)\varphi(q_1)]dS_q. \tag{3.2''}$$

We use the regular representations (3.1) and (3.2) for realization of the recurrence relations (2.19). This gives

$$\varphi_n(q_1) = -\varphi_{n-1}(q_1) + \int_S \Gamma_2(q_1, q)[\varphi_{n-1}(q) - \varphi_{n-1}(q_1)]dS_q, \tag{3.3}$$

$$\varphi_n(q_1) = -\varphi_{n-1}(q_1) + \int_S [\Gamma_1(q_1, q)\varphi_{n-1}(q) - \Gamma_2(q_1, q)\varphi_{n-1}(q_1)]dS_q. \tag{3.4}$$

Naturally, a realization of these relations requires discretization of the surface S, i.e. its splitting into small (elementary) regions S_j ($j = 1, 2, \ldots, N$). We specify a point q_{1j} in each of these surface elements and call these points the reference points Although these points can be situated anywhere in the elementary regions, it is advisable to have them in the central parts.

We shall carry out the realization of the relations (3.3) and (3.4) at the reference points. The easiest way to do so is to proceed from the cubature formulas constructed under the condition that within each elementary region, the density referred to its value at the reference point is constant. Then any term in Darboux's sum can be represented as the product of the integrand at the reference point and the area of the corresponding elementary region. In this case, the term which formally tends to infinity (more precisely, becomes indeterminate) when the points q and q_1 coincide, vanishes.

Let us now consider the second method of computing singular integrals. It is found that if the elementary domain is a plane polygon, the singular integral is determined in a closed form (it is assumed that the density is constant within this domain). It should be observed that in this case the domain excluded from consideration (in accordance with the definition of a singular integral) is a circle. Naturally, in order to apply the formula indicated above, we must first polygonalize the surface (if it is initially curvilinear). The result is the simplest if the domain is a rectangle and the reference point is chosen at the centre. It follows from the formula (1.29) that the jump in the limiting values of the stress operator is equal twice the density, while it follows from the symmetry conditions that its values from different sides are equal in magnitude but opposite in sign (hence the limiting value of the stress vector is equal to the density if we take the sign into consideration). Such a method helps not only in finding the integral itself, but also its sum, including the term outside the integral sign.

In order to develop this approach, it is frequently recommended to construct more accurate cubature formulas [98, 99] and not to consider the density as constant within an elementary domain. In this case, naturally, the formulas become considerably more cumbersome.

Concerning the complexity of any cubature formulas, it should be noted that their construction must be associated not only with the practically attainable degree of accuracy, but also with the simplicity of the computer programming and the ease of programme debugging. By this we mean the following. Suppose that calculations have been made for a certain discretization. It may follow from an analysis of the obtained results that a finer discretization is required in certain domains. The question arises as to how much effort is required to carry out the necessary alteration in the preliminary information prepared for programming. It is desirable that this extra amount of work should be minimum.

Let us now go over directly to the methods of solving integral equations. A realization of the method of mechanical quadratures leads to a system of algebraic equations of the order $3N$ (where N is the number of elementary regions). In the case of a problem II^+, the system will be degenerate, and this requires the application of special refined methods for its solution. It should be also observed that the question about the convergence of the method of mechanical quadratures remains unanswered, since it is necessary to prove that upon increasing the number N, the approximate solution obtained (in the piecewise constant representation) tends to the exact solution.

The method of successive approximations is preferable on account of the following reasons. Firstly, it follows from the proven convergence of this method that its approximate realization leads to the exact solution, since for any finite sum of the series, the problem is reduced to the calculation of a finite number of improper integrals, which can be always realized with any degree of precision.[16] Secondly, the very realization of the method with the help of a computer requires that only two iterations be preserved in the rapid storage (i.e. $6N$ numbers, while in the method of mechanical quadratures it is necessary to store $9N^2$ numbers).

Let us consider one more situation. Calculations have shown (this circumstance is in accordance with the results following from the resolvent theory) that starting from a certain value of n, the functions $\varphi_n(q)$ behave at all points as terms of a geometric progression with the same common ratio (determined by the position of the second resolvent pole). Hence, if the degree of convergence is not high, it is possible, after establishing a reliable value for the ratio, to obtain the analytic sum of the progression and then obtain the exact value of density.

Let us consider a method with the help of which we can considerably cut down the amount of calculations while maintaining the same degree of accuracy [100]. Suppose that a certain discretization of the surface has been introduced and that the reference points are given. We shall calculate all the iterations with the help of

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[16] The questions concerning the accuracy of calculations are not considered here.

cubatures (3.3) and (3.4) only for some of the reference points. For the remaining ones, we shall carry out interpolation with the help of known values at those points where cubatures were employed. The separation of points into two such groups must be carried out from an analysis of preliminary calculations, and in the domains where the functions $\varphi_n(q)$ change insignificantly interpolation should be carried out, since a fine discretization ensures a high degree of accuracy of cubatures.

A different method has been described in [101] for increasing the effectiveness of the algorithm. A net with a "varying" step is used. By this we mean that the calculation of functions $\varphi_n(q)$ at a certain point is carried out with the help of one (general) discretization of the surface and a local (in the vicinity of the given point) discretization which, naturally, is finer. In this case, the functions $\varphi_{n-1}(q)$ at the additionally created reference points are obtained with the help of interpolation. An essentially similar idea has been put forth in a somewhat different form in [102]. While calculating the cubatures, it is proposed that each term of integral sums be determined in a different way, depending on the distance between the point at which the integral is calculated and the corresponding elementary domain defining the term of the sum. For large distances, we can use the simplest formula in which the integrand at the reference point is multiplied by the area of the domain. For small distances, Gauss' cubature formulas should be used, whose order increases with decreasing distance. The values of the functions $\varphi_{n-1}(q)$ at the internal points of elementary domains, which must be known for realization of Gauss' formulas, are also obtained by interpolation. The authors of this work have also given recommendations regarding the choice of the order of Gauss' formulas depending on the indicated distance. These recommendations are based on model calculations for a plane domain broken into squares. Naturally, for a sufficiently fine discretization (compared with the radii of curvature of the surface), these recommendations may be applied to any type of surface.

The same work [102] contains considerations for constructing a fairly effective programme for a wide class of domains when it is quite cumbersome to prepare the initial data containing an analysis of different versions of surface discretization, leading to the construction of a solution with maximum accuracy. It is proposed that the entire surface be divided into fairly big parts, each of which, generally speaking, is a curvilinear quadrangle. With the help of an appropriate curvilinear system of coordinates u, v, each of these quadrangles is mapped into a square with side 2, having its centre at the origin of coordinates. Approximately, such a system of coordinates may be introduced with the help of the so-called form functions [103]. Satisfactory results have been obtained with the help of second-order form functions. Suppose that the vertices x_1, x_2, x_3, and x_4 of a rectangle are transformed into the vertices of a square (i.e. into points $u_1 = 1$, $v_1 = -1$, $u_2 = 1$, $v_2 = 1$, etc.), the point x_5 on the side x_1x_2 into the point $u_5 = 1$, $v_5 = 0$, the point x_6 on the side x_3x_4 into the point $u_6 = -1$, $v_6 = 0$, the point x_7 on the side x_2x_3 into the point $u_7 = 0$, $v_7 = 1$, and the point x_8 on the side x_4x_1 into the point $u_8 = 0$, $v_8 = -1$.

The second-order form functions are given by

$$N_i(u, v) = \frac{1}{4}(1 + uu_i)(1 + vv_i)(uu_i + vv_i - 1) \qquad (i = 1, 2, 3, 4),$$

$$N_i(u, v) = \frac{1}{2}(1 - v^2)(1 + uu_i) \qquad (i = 5, 6),$$

$$N_i(u, v) = \frac{1}{2}(1 - u^2)(1 + vv_i) \qquad (i = 7, 8).$$

Then, for the quadrangle surface, we get the approximate representation

$$\mathbf{x}(u, v) = \sum_{i=1}^{8} N_i(u, v)\mathbf{x}_i \quad (|u, v| \leqslant 1). \tag{3.5}$$

The second-order form functions lead to the construction of a total continuous surface, if the points on the adjacent sides are taken to be the same. In order to construct a smooth representation for the surface, we must turn to higher-order form functions.

The discretization of each quadrangle is carried out by specifying several points on two adjacent sides of a square in the (u, v)-plane. Segments parallel to the sides are drawn through these points. After this, we just have to use the representation (3.5).

Naturally, the division of a surface into large quadrangles and their division, in turn, into smaller ones (their number and size) are determined by the surface itself, as well as by the nature of the required potential density. First of all, it is essential that the representation (3.5) should satisfactorily approximate the surface. Secondly, the regions of anticipated sharp changes in potential, requiring a fine discretization locally, should be isolated into separate quadrangles during primary disretization. Otherwise, in view of the structure of the algorithm (ensuring an automation of the secondary discretization), we shall have to introduce unduly fine discretization of surface on an adjacent domain, thus causing an overall decrease in the potentialities of the computation programme.

It is clear that the concepts described above can be used in the case of the algorithm when the calculation of iterations at individual points is carried out by means of interpolation.

On the basis of the considerations given here (within the framework of a unified programme), the solution of the following three-dimensional problem is also given in [102]. An elastic body formed by an intersection of identical hollow cylinders is shown in Fig. 70. For calculations, it was assumed that $H/r = 6$, $R/r = 2$, $\sigma = 0.3$. The loading was carried out by hydrostatic pressure of unit magnitude applied to the inner surface. In view of the existence of three symmetry planes, iterations were calculated on 1/16 part of the surface (with appropriate recalculation of the functions $\varphi_n(q)$ on the remaining parts of the surface, since some components reversed their sign). Figure 71 shows three quadrangles into which the part of the

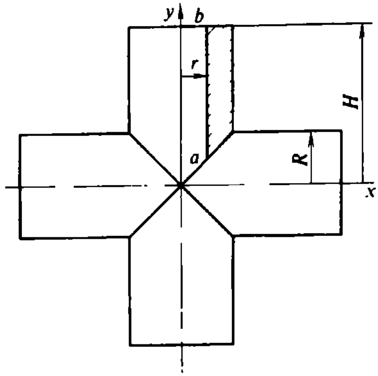

Fig. 70. Combination of two hollow cylinders.

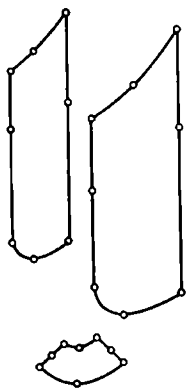

Fig. 71. Splitting of a surface into quadrangular domains.

surface under consideration was divided, as well as the position of those points on sides which, together with the vertices, were used for constructing the form functions. The discretization was carried out by dividing inner surface into 14×8 parts, the outer surface into 6×6 parts, and the end face into 6×4 parts. In all, seven iterations were carried out. Figure 72 gives the values of stresses on the line *ab*. It follows from the calculations that the solution is transformed into Lamé solution at distances $y > 3r$ from the joint.

Let us now consider the solution of axially symmetric problems, starting from the equations of the three-dimensional problem. In this case, it is natural to start from a discretization determined by parallels and meridians. Apparently, the vector $\varphi(q)$

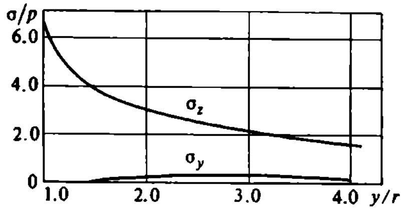

Fig. 72. The stress diagram.

will be constant at each parallel if we express it in a cylindrical system of coordinates (in the projections φ_ρ and φ_z). In the Cartesian coordinates (i.e. such coordinates that are used in the equations considered above), the following relations hold:

$$\varphi_x = \varphi_\rho \cos\varphi, \ \varphi_y = \varphi_\rho \sin\varphi, \ \varphi_z = \varphi_z.$$

These relations should be used in the recurrence relations (2.19). The counting procedure should be organized in such a way that the Darboux sum could be successively calculated along each parallel. In this case, it is useful to obtain the numerical [100] or the analytic [104] sum of the corresponding combinations of the integrands (multiplied by $\cos\varphi$ and $\sin\varphi$). As a result, the amount of information necessary for repeating the iterations is found to be quite compact and can be stored in the memory of the computer. Hence, the carrying out of each subsequent iteration will consume much less time as compared to the first iteration. Naturally, the calculations for the density should be carried out at the points of only one (initial) meridian.

We now go over to the effect of the errors in cubature formulas on the convergence of an algorithm. It is obvious that in the case of problems I^+ and II^-, this error (if it is sufficiently small) will not affect the convergence (if we use the formulas (2.31′), of course), but will lead to a certain error in the solution (similar to the retention of a finite sum in the series). A quantitative estimate of the permissible error in the computational programme (guaranteeing convergence) may be expressed in terms of the effective change in the value of the parameter λ (assuming that there are no other errors): the value of λ must not exceed modulo the second-largest resolvent pole.

In the case of the problem I^- (assuming that the eigenfunctions of the companion equation have already been determined and the condition of the existence of a solution under the given boundary condition has been verified) and the problem II^+, we should use the concepts described in Sec. 2, Ch. 1, Vol. 1. The first method states that the series (2.18) should be considered in the asymptotic sense, without carrying out an indefinitely large number of iterations for a fixed discretization of the surface. The second method involves the correction of each iteration (carrying out the orthogonal projection onto the subspace of functions satisfying the orthogonality condition). Then, formula (2.32′), Ch. 1, Vol. 1 is transformed as follows:

$$\tilde{\varphi}_n(q) = \varphi_n(q) - \sum_{i=1}^{6} \varphi_{1i}(q) \int_S \varphi_{1i}(q_1)\varphi_n(q_1)dS_{q_1}, \tag{3.5'}$$

where $\varphi_{1i}(q)$ are eigenfunctions of the companion equations (i.e. the equations of the problem I), taken in the orthonormal form. While constructing the functions, we can proceed from the following orthogonal system of linear functions:

$$\varphi_{11}^* = (1, 0, 0), \varphi_{12}^* = (0, 1, 0), \varphi_{13}^* = (0, 0, 1),$$

$$\varphi_{14}^* = (x_3, 0, -x_1), \varphi_{15}^* = (-x_2, x_1, 0), \varphi_{16}^* = (0, -x_3, x_2).$$

The normalization of these functions can be carried out without any difficulty.

The above concepts of the effect of the computational errors on the convergence for the case of axial symmetry are, of course, simplified, and the formula (3.5′) assumes the form

$$\tilde{\varphi}_n(q) = \varphi_n(q) - \nu' \int_S \nu' \varphi_n(q_1)dS_{q_1}, \tag{3.6}$$

where the vector ν' is directed along the axis of rotation and is equal to $1/\sqrt{S}$ in magnitude, S being the surface area.

It should be noted that if integration over the angle is carried out numerically, it is naturally required to specify a definite discretization over the angle. In this case, it is necessary that this discretization should not violate the symmetry with respect to two planes. If the zeroth meridian is defined by the angle $\varphi = 0$, the discretization must be symmetric with respect to $\varphi = 0$ and $\varphi = \pi/2$. If this is not so, we must use the general formula (3.5′).

Let us approach this question from a different angle. It follows from formulas (2.22) that the iterative process for an integral equation of the problem II$^+$ leads to an eigenfunction corresponding to the point -1 (if the load is not self-balanced, of course). The sum of the iterative series will be divergent, while the solution in terms of stresses will converge. In the present case, the load is self-balancing according to the formulation of the boundary value problem, although divergence occurs as a result of error in the computation programme (except for the case when we have three planes of geometrical and force symmetry). It is logical to treat this error as an error introduced into the boundary conditions (thus violating the requirement of their self-balancing) and followed by an exact realization. It can then be stated that for a sufficiently fine discretization of the surface, the process converges in terms of stresses, which fully satisfies the formulation of the problem. Naturally, the finer the discretization, the closer the solution obtained to the exact one.

In order to illustrate these points, the axially symmetric problem for a cylinder with rounded ends was solved. The loading was reduced to shearing stresses applied to two strips on the cylindrical surface, one strip twice as wide as the other. The stresses on each strip were distributed linearly, from zero to the maximum and back to zero. The maximum value of stress on the wide strip was equal to unity, while on the narrow strip, it was equal to -2. Thus, the condition that the principal vector be equal to zero was satisfied. Up to 70 reference points were taken along the con-

TABLE 7

	φ_4	φ_5	φ_7	φ_8	φ_{14}	φ_{15}
I	7.9-04	−5.1-03	−7.0-03	7.9-03	1.6-02	−1.7-02
II	−3.2-03	−7.9-04	−1.7-03	2.0-03	3.9-03	−4.3-03
III	−4.1-03	2.4-04	−3.5-04	4.5-04	9.6-04	−1.1-03

	φ^5	φ^8	φ^{15}	σ_z^5	σ_z^8	σ_z^{15}
I	−8.58-01	−8.38-01	−7.48-01	−1.918-01	−1.925-01	−1.927-01
II	−8.78-01	−8.73-01	−8.51-01	−2.060-01	−2.067-01	−2.068-01
III	−8.82-01	−8.82-01	−8.76-01	−2.095-01	−2.1027-01	−2.2039-01

tour, most of them in the regions where the load was applied. Calculations were made for three discretizations differing in the number of reference points near the middle of the narrow strip (where the shearing stresses are the highest). The minimum number of points was used for the first calculation. For the second, four more points were taken, while another two were added for the third. Table 7 gives the values of the zth component of the functions $\varphi_n(q)$ at one of the reference points (on the narrow strip) for different values of n, corresponding to the sum of the series $\varphi^N(q)$, as well as the values of the component σ_z (the z-axis is the axis of rotation), at one of the internal points in the vicinity of the loading. All results have been directly obtained with the help of formula (3.4).

It follows from the data given in the table that for a fixed number of iterations, the increased accuracy in calculations (transition from the first computation to the third) leads to convergence of the iterative sum, thus causing the convergence of stresses. If we proceed from a fixed discretization, the functions $\varphi_n(q)$ start to increase after a certain number, thus leading to a divergence of the algorithm. Significantly, the functions $\varphi_n(q)$ themselves decrease in this case with increasing accuracy of calculations. The fact that the stresses are convergent even for a diverging solution of an integral equation is completely in accord with the statement made above.

Calculations were also made by using the correction in accordance with (3.6). All these results are given in Table 8.

Here, the convergence of all the iterations and the convergence of sums are precisely taken into consideration. The solution in terms of stresses, naturally, remains convergent. It should be noted that for the same discretization, the values of stresses obtained by using a correction are quite close to those obtained without correction.

TABLE 8

	φ_4	φ_5	φ_7	φ_8	φ_{14}	φ_{15}
I	−4.2-04	5.4-04	6.6-05	−3.4-05	−2.3-07	1.15-07
II	−4.6-03	6.1-04	7.8-05	−4.0-05	−2.5-07	1.25-07
III	−4.5-03	6.3-04	8.0-05	−4.0-05	−2.5-07	1.27-07

	φ^5	φ^8	φ^{15}	σ_z^5	σ_z^8	σ_z^{15}
I	−8.82-01	−8.8231-01	−8.8233-01	−1.918-01	−1.9247-01	−1.926-01
II	−8.84-01	−8.8467-01	−8.8469-01	−2.059-01	−2.0670-01	−2.068-01
III	−8.84-01	−8.844-01	−8.8455-01	−2.059-01	−2.1027-01	−2.104-01

The body under consideration has a symmetry plane. Hence the eigenfunction also must be symmetric relative to this plane (the five remaining functions are cancelled on account of axial symmetry). In view of the statement that the process gives fairly accurately the eigenfunction on account of errors in the computation programme, it could be expected that for a sufficiently large number, the functions $\varphi_n(q)$ will have the given kind of symmetry. The calculations fully confirm this view.

Let us now calculate the stresses and displacements after a direct solution of the integral equation has been obtained. Strictly speaking, we should consider the calculation of stresses at the points on the boundary surface, since the calculation of stresses and displacements at the internal points of the domain is reduced to solving the integrals with analytic kernels, while the computation of displacements at the points on the surface is reduced to solving improper integrals[17], for which standard methods may be employed. True, it should be noted that when in this process calculations are carried out at points lying in the vicinity of the boundary, we must introduce a secondary discretization of the surface in the region lying in the neighbourhood of the point under consideration. In this case, the value of the density used in calculations should be obtained from some interpolation, starting from the solution of the integral equation. The required values of stresses and displacements may be assumed to be determined with a sufficient degree of accuracy (depending on the degree of accuracy of the solution of the integral equation) only when there are no changes in the required values as a result of a secondary (even finer) discretization.

[17] While using the integral equation (2.5), the displacements at the surface are obtained from the solution of the integral equation.

Generally speaking, once the displacements have been determined at a few closely situated points on the boundary surface, we may determine, by calculating the strains from known values of the stress vector (which are given in the formulation of the problem), the remaining components of the stress tensor also.

We shall now describe another method whose advantage lies in the fact that its realization does not require the construction of any additional blocks in the programme used for the integral equation. We consider a few points in the neighbourhood of a given point on the surface (for the sake of convenience, these points are situated on the normal to the surface), and calculate all the components of the stress tensor at these points. This procedure may be accomplished with the help of a block of the basic programme, ensuring the computation of the integral

$$\int_S \mathbf{\Gamma}_1(q_1, q)\varphi(q)\, dS_q$$

by specifying successively three directions of the normal. Further, with the help of the secondary discretization, we should try to ensure that the values of the stresses obtained should be stable. At the final stage, some kind of extrapolation (for example, polynomial) should be carried out at the given boundary point.

It should be noted that with this kind of approach, the situation is most favourable for determining the components appearing in the boundary conditions[18]. This circumstance may be used for determining the accuracy of extrapolation.

In the case of axial symmetry, the problem of determining the stresses at the boundary is reduced (by carrying out analytic integration over the angle) to the calculation of one-dimensional singular integrals, which may be carried out without any difficulty (see Sec. 3, Ch. 1, Vol. 1).

Naturally, while using any method of calculations, we would like to estimate the accuracy of the solution obtained. Besides the traditional methods (calculations for a finer discretization, error estimation in the boundary conditions, the method based on the fact that for $\sigma = 0.5$, Eq. (2.3) has a known eigenfunction, viz. the vector function directed along the normal to the surface and having a modulus equal to unity. Hence, it is proposed that the iterative process should be carried out for $\sigma = 0.5$ for a given discretization and under the same boundary conditions. Deviations in the above-mentioned conditions for the functions $\varphi_n(q)$ may also serve as an estimate of the error in the solution of the original problem.

Let us consider a method which significantly increases the accuracy for a number of problems. There exist certain problems in the theory of elasticity, whose solution is expressed quite accurately but the calculations by the potential theory are quite cumbersome. An example of this kind of problems is that of a thin-walled hollow sphere. Hence, while constructing the solution of some problems, it may be useful to superimpose a certain specially selected solution so that the modified boundary value problem could be solved fairly effectively by the potential theory. This

[18] There is no need to calculate these by the method indicated here. The accuracy with which the given boundary conditions are satisfied is directly defined by the integral equations using regular representations.

method has been illustrated in [105] on the basis of the axially symmetric problem for a cylinder with a spherical cavity, when the ratio of the radii of the cylinder and the cavity is 1.1. The loading was reduced to hydrostatic pressure on the sphere. The solution for a hollow sphere with outer radius equal to the radius of the cylinder and inner radius equal to the radius of the cavity was superimposed. As a result, the load on the sphere vanished,while certain stresses appeared on the cylindrical surface and the lateral surface. Significantly, the principal stress vector in the most dangerous cross section is now equal to zero.

The above discussion refers to the realization of solution of integral equations corresponding to the basic problems in the theory of elasticity, when the boundary surface is sufficiently smooth. Let us consider the case when the surface is piecewise smooth, i.e. consists of open smooth surfaces having common boundaries along certain lines which in turn may have corner points. Inside each surface of this kind, some sort of a boundary condition is given, while the boundary conditions at the edges or the corner points should be treated as limiting conditions from the side of some surface. It is assumed that concentrated forces are not applied at irregular points[19].

Starting, as before, from the generalized elastic potentials, we can easily construct integral equations for this kind of domain. However, in this case the kernels and the coefficient of the term outside the integral sign undergo discontinuities. Unfortunately, there is no theory for this kind of equations, but we can easily extend the computational programmes described above (although in a somewhat more complicated form).

Let us separately discretize each of the surfaces. In this way, no elementary domain will be bounded by several surfaces, and the regular points will belong to two (or more) surfaces. Under such a discretization, all the central points will be the regular points of the surface, and hence there is no need to reconstruct the computational formulas. We proceed in a similar way for the case of isolated singular points (conical points): each of these points must be the apex of several elementary domains.

Let us suppose that a decrease in the size of elementary domains leads to a stabilization of densities and stresses everywhere except the immediate neighbourhood of irregular points, a region which also decreases with decreasing size of elementary domains. We can then speak about a satisfactory solution of the corresponding boundary value problem, if the stable values obtained for the stresses in the vicinity of irregular points (with the exception of the small domain mentioned above) will asymptotically approach the solutions given by Eqs. (8.34), (8.35), (8.52), and (8.53), Ch. 3, Vol. 1 for the case when the boundary conditions are consistent. Otherwise, the asymptote will be determined from an analysis of the solutions for the wedge-like domains.

Let us illustrate this by taking the example of an axially symmetric problem (external as well as internal) when the boundary surface is formed by the rotation of a

[19] Otherwise, we should eliminate them by the superposition of appropriate particular solutions.

TABLE 9

A	0.001	0.002	0.002	0.05
B	0.00002	0.00004	0.0003	0.00075
C	0.00001	0.00002	0.00003	0.00004

TABLE 10

I	0.99	0.982	0.96	0.95
II	0.9999	0.9997	0.99925	0.994
III	0.99995	0.9999	0.9998	0.9997

TABLE 11

z	0.9999			0.994			0.99			0.03		
	I	II	III	I	II	III	I	II	III	I	II	III
A	—	1.57	1.75	—	0.77	0.79	0.68	0.75	0.75	3.10	3.00	3.00
B	—	1.55	1.73	—	0.76	0.79	0.68	0.75	0.75	3.10	2.86	2.86
C	—	1.56	1.75	—	0.76	0.79	0.65	0.75	0.75	3.15	3.09	3.08

z	0.01			0.006			0.0001		
	I	II	III	I	II	III	I	II	III
A	4.88	4.42	4.40	—	5.30	5.25	—	4.48	4.65
B	4.43	4.11	4.08	—	5.00	4.64	—	19.4	21.7
C	5.09	4.70	4.64	—	5.81	5.67	—	26.09	29.20

square of side $\sqrt{2}$ around its diagonal, and the load has been reduced to hydrostatic pressure of unit magnitude. While solving this problem, numerical integration was carried out over the angle of rotation [100] and, hence, in order to estimate the effect of elementary domains on the accuracy of solution, the angle was divided in

three different ways — A, B, and C. Table 9 gives the values of four angles (in fractions of π) between the meridians (forming surface discretization) close to zeroth meridian (at which all calculations were carried out).

Different versions of positions of reference points on the cross section contour were also specified. On the sides of the square in the interval $0.01 \leqslant |z| \leqslant 0.99$ (here, the z-coordinate of the points is given) 27 points were always specified, situated symmetrically with respect to the midpoint of the side of the square. The reference points were added (also symmetrically) only in the regions adjoining a vertex or a side. Table 10 gives the values of the coordinate of four reference points lying close to the vertex for three versions — I, II, and III.

Calculations were carried out for nine versions of surface discretization, obtained from different possible combinations of the division of the angle of rotation and of the contour, as mentioned above. Table 11 gives the values of the z-component of the density at several reference points (their z-coordinate is given) for all the nine versions of discretization.

It follows from the data given above that a decrease in the size of the elementary domains leads to the attainment of stable values of density.

Calculations for stresses have shown that for the internal problem, the state of stress is determined over the entire body with an error smaller than 0.005, except for the region adjoining a vertex or a side at distances smaller than 0.05. Divergence in stress values for different discretizations of the solution of the integral equation was observed only at very short distances from a vertex or a side (of the order of 0.001).

The solution of the external problem led to different results in stresses only at points situated in the vicinity of the same irregular points on the surface (at distances less than 0.03).

We shall consider the case of the external problem in detail, since the solid angles at the apex and the edge are such that the corresponding equations (8.53) and (8.34), Ch. 3, Vol. 1 will have non-trivial solutions ($\mathrm{Re}\,\lambda < 1$), and hence the stresses will have a singularity. It follows from Eq. (8.33), Ch. 3, Vol. 1 (see Fig. 25) that for $\nu = 0.3$, the stresses in the vicinity of the apex will have the asymptotics $C\rho^{-0.2}$ in local spherical coordinates. On the other hand, it follows from Eq. (8.34), Ch. 3, Vol. 1 (see Fig. 22) that the stresses in the neighbourhood of the edge will have the asymptotics $C\rho^{-0.455}$ (in local polar coordinates in some meridianal plane). Naturally, the constants in the example given above will be different for each component of the stress tensor.

We shall now describe a method which can be used for determining the values of these constants, proceeding from the numerical solution of the problem, valid in the vicinity of singular points (although at a certain distance from them). We shall start with the seemingly natural requirement that the numerical solution should turn into the asymptotic solution with a continuous contact. For this purpose, it is sufficient to determine the values of some stress component at a number of points near the singular point and lying, say, on a straight line passing through the singular point. These values should be determined both numerically and analytically, starting from the representation of $\rho^{\lambda-1}$ (with an appropriate value of λ) in the localized system

TABLE 12

r	σ_r	σ_φ	σ_z	$\rho^{-0.455}$	$\sigma_r\rho^{-0.455}$	$\sigma_\varphi\rho^{-0.455}$	$\sigma_z\rho^{-0.455}$
1.00003	25.10	19.76	37.54	114.26	0.220	0.173	0.328
1.00005	21.21	15.72	30.43	90.57	0.234	0.174	0.336
1.00008	17.25	12.06	25.05	73.13	0.235	0.176	0.342
1.00012	14.21	10.78	21.11	60.80	0.233	0.177	0.347
1.00015	12.73	9.78	19.20	54.94	0.231	0.178	0.349
1.00035	8.39	6.81	13.42	37.36	0.225	0.182	0.359
1.00100	5.05	4.29	8.52	23.17	0.218	0.185	0.368
1.00250	3.16	2.81	5.56	15.27	0.207	0.183	0.364
1.00400	2.43	2.23	4.41	12.33	0.197	0.181	0.358

TABLE 13

z	$\sigma_r = \sigma_\varphi$	σ_z	$\rho^{-0.2}$	$\sigma_z\rho^{-0.2}$	$\sigma_z\rho^{-0.2}$
1.000001	8.29	6.44	15.85	0.522	0.406
1.000003	7.74	6.26	12.72	0.609	0.500
1.000007	6.90	5.83	10.74	0.642	0.543
1.000010	6.43	5.49	10.00	0.643	0.549
1.000015	5.87	4.15	9.22	0.636	0.538
1.000025	5.13	4.07	8.32	0.615	0.489

TABLE 14

		C_r	C_φ	C_z	Δ_1	Δ_2	C'_r	C'_z
I	*A*	0.20	0.18	0.34	0.09	0:15	0.52	0.43
	B	0.16	0.16	0.32	0.09	0.35	0.51	0.45
	C	0.21	0.16	0.34	0.09	0.10	0.51	0.46
II	*A*	0.11	0.16	0.31	0.23	0.30	0.60	0.43
	B	0.16	0.16	0.33	0.20	0.40	0.58	0.42
	C	0.22	0.19	0.38	0.05	0.16	0.59	0.43
III	*A*	0.13	0.16	0.33	0.16	0.70	0.65	0.55
	B	0.18	0.16	0.32	0.06	0.24	0.64	0.54
	C	0.23	0.18	0.37	0.02	0.06	0.64	0.55

of polar or spherical coordinates. After this, the ratio of specified quantities is determined. The point (on the straight line under consideration) at which the numerical solution turns into asymptotic solution is taken to be the point at which this ratio has an extremal value. The ratio at this point itself is taken to be the required value of the constant for the corresponding stress component.

Table 12 gives the values obtained as a result of the calculation IIIC of the stress components at points on the symmetry plane in the cylindrical system of coordinates, the values of the function $\rho^{-0.455}$, and the ratios of stresses to this function.

It follows from the data given here that the maximum is considered quite accurately. On the basis of these data, we can propose the following values of constants for the asymptotics $\sigma_r = C_r\rho^{-0.455}$, $\sigma_\varphi = C_\varphi\rho^{-0.455}$, and $\sigma_z = C_z\rho^{-0.455}$ ($\rho = |r| - 1$): $C_r = 0.235$, $C_\varphi = 0.185$, and $C_z = 0.368$. The asymptotics constructed in this way should be used in the vicinity of the side at distances less than 0.0001.

Figure 73 shows the components σ_z obtained by solving the integral equation (points) as well as by the asymptotic representation (solid line).

Table 13 gives the values of stresses $\sigma_r = \sigma_\varphi$, and σ_z on the rotational axis, as well as the values of the function and the ratios of stresses to this function $\rho^{-0.2}$.

The maximum is overlooked in this case as well. Thus, the following values of constants can be proposed for the asymptotics $\sigma_r = C_r'\rho^{-0.2}$ and $\sigma_z = C_z'\rho^{-0.2}$ ($\rho = |z| - 1$): $C_r = 0.643$ and $C_z' = 0.549$. Figure 74 shows the values of the compo-

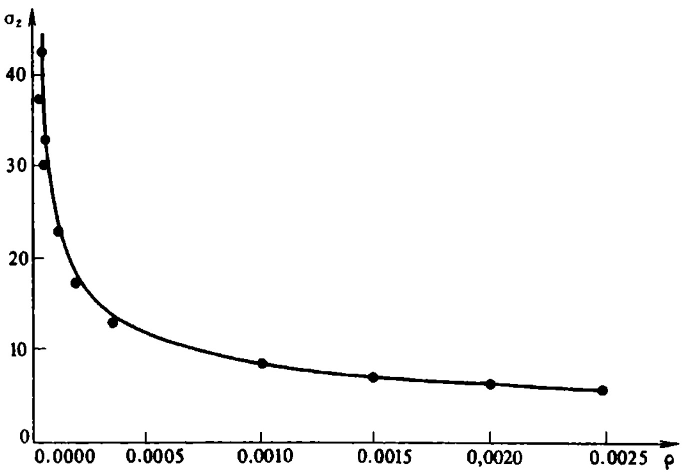

Fig. 73. Stress σ_z on the midplane.

nent σ_r obtained by solving the integral equation (points) as well as by the asymptotic representation (solid line).

It is possible to obtain an exact relation between the coefficients of the asymptotics, which should be used for estimation of error in their approximate determination. It was mentioned in Sec. 8, Ch. 3, Vol. 1 that the asymptotics of the solution near an edge is exactly the same as in the case of plane strain. The relation (4.5) in Ch. 3, Vol. 1 is satisfied for stresses in the case of plane strain. Hence the relation $C_\varphi = \sigma(C_r + C_z)$ must be satisfied for the coefficients C_r, C_φ, and C_z.

On the other hand, since the problem is symmetric relative to the symmetry plane, the asymptotic series appearing in the solution near an edge must contain only symmetric solutions. Hence in the asymptotic series used for constructing the terms, the constants B and D in Eq. (8.24), Ch. 3, Vol. 1 should be put equal to zero. The coefficients A and C will satisfy the relation (8.31), Ch. 3, Vol. 1, which leads to a relation between the coefficients C_r and C_z. In the present case, for $\alpha = (3/4)\pi$ and $\lambda = 0.545$, we get $C_r/C_z = 1.473$. Note that this ratio does not depend on Poisson's ratio.

Table 14 gives the values of coefficients C_r, C_φ, C_z, C_r', and C_z' for all nine versions of calculations, as well as the values of the error introduced by the ratios given above. Naturally, it is possible, through more cumbersome relations, to obtain a relation between the coefficients C_r' and C_z'.

The problem was also solved for $\sigma = 0.5$. In this case, the iterations $\varphi_n(q)$ quite rapidly reach the vector function directed along the normal to the surface and having a constant modulus. Deviations of more than one degree in the angle and of 0.03

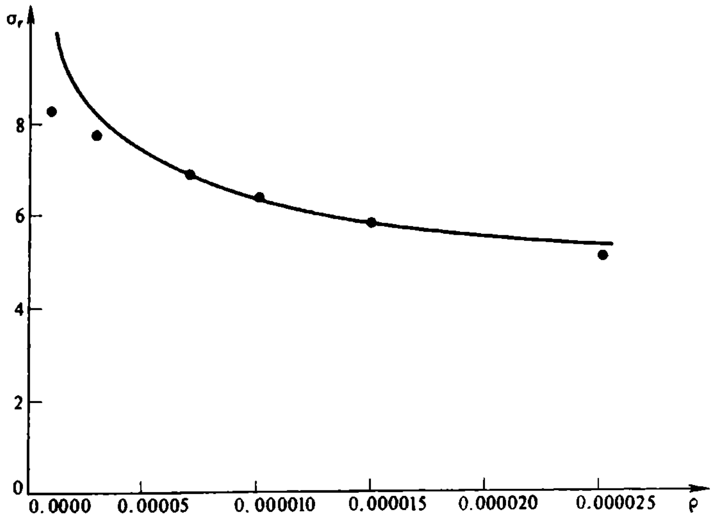

Fig. 74. Stress σ_r on rotational axis.

TABLE 15

r	τ_{rz}	$\rho^{-0.1}$	$\tau_{rz}\rho^{0.1}$
1.00001	1.093	3.16	0.346
1.00003	1.070	2.83	0.378
1.00005	0.995	2.67	0.373
1.00008	0.907	2.57	0.353

in magnitude were formed only in the regions near a vertex (distance shorter than 0.03) and a side (distance shorter than 0.12).

In conclusion, it should be mentioned that a solution was also constructed for the same region when the pressure on the lower and upper parts had different signs. It follows from symmetry considerations that normal stresses vanish on the symmetry plane, while the shearing stresses will have asymptotics given by the root of Eq. (3.32). This root is equal to 0.9 for $\alpha = (3/4)\pi$. Table 15 gives the values of shearing stresses τ_{rz} on the symmetry plane and the function $\rho^{-0.1}$, as well as their ratio.

In this case also, the maximum is overlooked. For the asymptotics, we get the representation $\tau_{rz} = 0.378\rho^{-0.1}$.

The method described above was used in [106] for solving the problem in which two identical cubic cavities are situated in an infinite space (Fig. 75). The loading was reduced to a hydrostatic pressure[20]. The discretization was carried out as

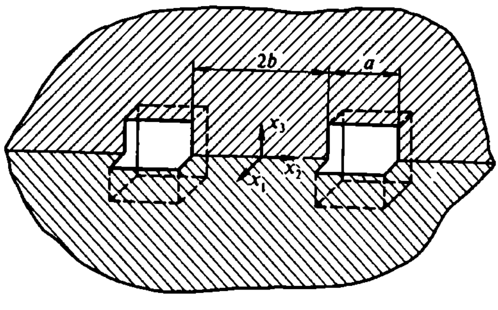

Fig. 75. A space with two cubic cavities.

[20] By superimposing a trivial solution, we can easily go over to the problem in which a hydrostatic stress is applied at infinity, while the cavities are free from load.

follows. On each edge, we specify several points condensed towards the vertices. The segments joining these points divide each edge of the cavities into rectangles of different sizes. The existence of three symmetry planes was used for reducing the amount of calculations: all the functions $\varphi_n(q)$ were determined only at a quarter of the surface of one of the cavities, and the values for the functions $\varphi_n(q)$ at the remaining points of this surface, as well as for all the points on the other surfaces, were determined from symmetry considerations. Figure 76 shows the curves for stresses $\sigma_1 = \sigma_3$ and σ_2 on the line joining the centres of the cavities for different thicknesses of the bridge ($2b = 3a$, and $2b = 0.5a$). In this case, seven points were chosen, dividing the edge into segments $0.05a$, $0.10a$, $0.15a$, $0.20a$, and so on, symmetrically. It was possible to resort to just seven iterations on account of $\varphi_n(q)$ tending rapidly to zero. It was shown that splitting the edge into 10 segments instead of seven practically does not change the values of the stresses.

As mentioned in Sec. 2, Eqs. (2.3) and (2.5) have equivalent spectral properties. Let us compare them from the point of view of the numerical solution. In the first case (Eq. (2.3)), the boundary condition is given in the statement of the problem, while in the second case (Eq. (2.5)), it has to be calculated first in a fairly dense set of points on the boundary surface. The procedure itself is quite cumbersome and involves the solution of two-dimensional improper integrals. True, the solution of the integral equation at once gives the displacements at the boundary points, while in Eq. (2.3), it is necessary for this purpose to calculate the same improper integrals after obtaining the density values. The basic difference lies in that, as a rule, even if it is required to determine the displacements at the boundary points from the formulation of the problem, it has to be done only at a few points. If, however, we speak of stresses at the internal or boundary points, the representation of displacements in the form of a single layer potential turns out to be more

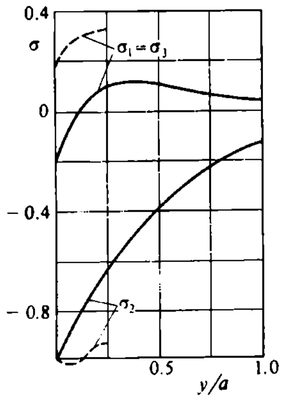

Fig. 76. Stresses σ_1, σ_2, and σ_3 along the line joining the centres of cubic cavities.

economical from the calculation point of view than the representations (1.12) or (1.14).

Another point of considerable importance should be noted here. Solving the integral equation (2.3) approximately and substituting the solution obtained into the left-hand side, we can judge the accuracy of the solution of the boundary value problem from the magnitude of the discrepancy in the right-hand side. In the case of Eq. (2.5), such a discrepancy will include the error introduced while calculating the right-hand side, and hence will determine the error in the solution of the boundary value problem to a lesser extent.

Let us consider another method of numerical solution of the three-dimensional problems in the theory of elasticity [107]. This is the method of direct solution of the functional equations obtained from the identities (1.13) and (1.15), when the displacements or the stresses at the surface are known (and thus the stresses or the displacements respectively are not known). In this case, some sort of discretization of the surface S is proposed, and the values of the stresses or displacements at the central points are given as unknown quantities. In order to determine their values outside the domain, a certain set of points (whose number equal to the number of elementary domains) is specified, and it is required to satisfy the identities (1.13) or (1.15) at these points. The questions of actual realization of this method (which essentially boil down to a suitable choice of these points) have been considered in [108], where it is also shown that if a polygonization of the surface is carried out, all integrals are calculated in a closed form.

Before concluding this section, let us consider the solution of singular integral equations in the problems of steady-state oscillations. In this case, the integrals may be evaluated by using representations similar to (3.1) or (3.2), or (if a polygonalization of the boundary surface has been carried out) by using the formulas obtained in [109].

If the oscillation frequency for the exterior and interior problems is less than the first natural frequency for the interior problem, the solution of the integral equation may be obtained by successive approximations. In the general case, however, it is expedient to use the method of mechanical quadratures, reducing the problem to a system of algebraic equations. This system enables us to determine the values of natural frequencies which are the roots of its determinant.

Section 4
Body Forces. The Temperature Problem

Since the equations in the theory of elasticity are linear, the problems involving body forces (of any origin) are reduced to the case where no body forces are present, by superimposing particular solutions. This statement is also valid while solving problems in the theory of elasticity involving stresses and displacements. The boundary conditions in this case are changed accordingly.

In the case of gravitational field, where the density is constant, the particular solution can be easily found:

$$\sigma_x = \sigma_y = 0, \quad \sigma_z = \rho g z, \tag{4.1}$$

where ρ is the density and g is the acceleration due to gravity. Here, it is assumed that the force of gravity acts along the z-axis. It should be noted that a more complex solution is adopted in rock mechanics:

$$\sigma_x = \sigma_y = k\sigma_z, \tag{4.2}$$

where k is a dimensionless constant which is determined experimentally. The presence of additional terms is explained by the fact that, in the prolonged course of rock formation, additional factors must be included to take into account the effect of time.

In the case of centrifugal forces (when the body rotates with an angular velocity ω), different versions of particular solution are possible. The simplest solution has the form

$$\begin{aligned} \sigma_x &= -\frac{2\lambda + \mu}{4(\lambda + 2\mu)}\rho\omega^2(x^2 + y^2) - \frac{\mu}{2(\lambda + 2\mu)}\rho\omega^2 x^2, \\ \sigma_y &= -\frac{2\lambda + \mu}{4(\lambda + 2\mu)}\rho\omega^2(x^2 + y^2) - \frac{\mu}{2(\lambda + 2\mu)}\rho\omega^2 y^2, \\ \tau_{xy} &= -\frac{\mu\rho\omega^2}{2(\lambda + 2\mu)}xy, \end{aligned} \tag{4.3}$$

where the z-axis coincides with the axis of rotation.

In the general case, the particular solution may be obtained with the help of the volume integral (1.25) (generalized Newtonian elastic potential):

$$\mathbf{U}(p) = \frac{1}{2}\int_{\Omega} \mathbf{\Gamma}(p, q)\varphi(q)d\Omega_q. \tag{4.4}$$

As was proved above, the displacements $\mathbf{U}(p)$ satisfy Lamé's equation with $\varphi(q)$ as the right-hand side.

In principle, it is not difficult to evaluate the integral (4.4) or the stresses corresponding to it,

$$\frac{1}{2}\int_{\Omega} T_{\nu(p)}\mathbf{\Gamma}(p, q)\varphi(q)d\Omega_q, \tag{4.5}$$

since the integrals are improper and we can apply the known cubature formulas, including the simplest ones, to them.

Obviously, the above discussion may be directly applied to problem of incoherent thermoelasticity which (as was indicated in Sec. 1, Ch. 3, Vol. 1) is reduced to the isothermal problem involving body forces γ grad T (see Eq. (1.21), Ch. 3, Vol. 1) and an additional load on the part of the boundary where the boundary conditions for stresses are given (the so-called temperature potential). These stresses are equal to γT and are directed along the normal to the surface.

Thus, the application of the generalized potential for the case of body forces leads to the following four stages in solving the auxiliary isothermal problem.

1. The construction of grad T and evaluation of the vector of stresses corresponding to the volume integral $\int_{\Omega} \mathbf{\Gamma}(p, q)$ grad $Td\Omega$ for a sufficiently dense set of boun-

dary points, as well as the construction of the stress tensor at given points lying in the volume and on the surface (at which it is required to determine the stresses).

2. The determination of modified boundary conditions when the boundary conditions given in the formulation of the problem are supplemented by stresses (with the opposite sign) generated by the volume potential, as well as the temperature potential.

3. The solution of the isothermal problem thus obtained, and the determination of the required displacements and stresses at given points.

4. The superimposition of the stresses generated by the volume potential on the obtained stresses. Transition to the actual stresses of the thermoelastic problem with the help of the terms γT for normal components of the stress tensor.

This method was applied in [108] for calculating the stresses in a cylinder with a cylindrical cavity.

It should be noted that the problem of thermoelasticity can be easily solved if we have a programme for solving the singular integral equation (2.3) by the method of successive approximations [109]. For this purpose, the entire volume should be divided into layers (within which the temperature gradients vary insignificantly). Each layer can be treated as a surface for which a solution of the problem in the theory of elasticity is obtained for the case when the boundary condition is equal to the temperature gradient multiplied by γ and the layer thickness. In this case, we must attain the zero of iterations, i.e. replace the potential density by the boundary condition and determine the stresses at the required points. An example of this kind is given in [109].

Several authors [110, 111] have shown that when there is no heat source (the temperature distribution then satisfies Laplace's equation), it is possible to replace volume integrals by surface integrals.

Section 5
Integral Equations for Mixed Problems

We shall assume that the surface S, bounding an elastic body, consists of two parts S_1 and S_2 which have a common boundary[21]. In the domain D, it is required to find the displacement vector $\mathbf{u}(p)$ proceeding from the boundary conditions

$$\begin{aligned} \lim_{p \to q} \mathbf{u}(p) &= \mathbf{F}_1(q) \quad (q \in S_1) \\ \lim_{p \to q} T_{\nu(p)}\mathbf{u}(p) &= \mathbf{F}_2(q) \quad (q \in S_2). \end{aligned} \tag{5.1}$$

By using some representation for displacements in the form of generalized elastic potentials, we can at once obtain specific equations for their densities with the help of Eqs. (5.1). For example, if we use a single layer potential $\mathbf{V}(p, \varphi)$, we get integral equations of the form

[21] In turn, S_1 and S_2 may be sums of a finite number of unclosed surfaces.

$$\int_S \Gamma(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}_1(q) \qquad (q \in S_1),$$
$$\varphi(q) - \lambda\int_S \Gamma_1(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}_2(q) \qquad (q \in S_2), \tag{5.2}$$

where the exterior problem corresponds to the parameter $\lambda = 1$, and the interior problem, to the parameter $\lambda = -1$. In the latter case, we must change the sign of the function $\mathbf{F}_2(q)$. This equation is an equation with a discontinuous kernel and a discontinuous coefficient of the term outside the integral sign. There is at present no theory for this kind of integral equations.

If, however, we use the representation for displacements in the form of a double layer potential $\mathbf{W}(p, \varphi)$, we obtain equations in the following form:

$$\varphi(q) - \lambda\int_S \Gamma_2(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}_1(q) \; (q \in S_1),$$
$$\lim_{p \to q} \; T_{\nu(p)}\int_S \Gamma_2(p, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}_2(q) \; (q \in S_2). \tag{5.3}$$

This equation cannot be classified as an integral equation, since we cannot change the order of differentiation and integration in the second equation.

Another method of constructing integral equations for the mixed problem is based on the use of two potentials whose densities are given on the unclosed surfaces S_1 and S_2:

$$\mathbf{W}(p, \varphi) = \int_{S_1} \Gamma_2(p, q)\varphi(q)dS_q \quad (q \in S_1),$$
$$\mathbf{V}(p, \varphi) = \int_{S_2} \Gamma(p, q)\varphi(q)dS_q \quad (q \in S_2). \tag{5.4}$$

Carrying out the limit transition to the boundary for the sum $\mathbf{V} + \mathbf{W}$, we get a system of singular equations with discontinuous kernels for the internal points of the surfaces S_1 and S_2:

$$\varphi(q) - \lambda\int_{S_1} \Gamma_2(q, q_1)\varphi(q_1)\, dS_{q_1}$$
$$- \lambda\int_{S_2} \Gamma\,(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}_1(q) \quad (q \in S_1),$$
$$\varphi(q) = \lambda\int_{S_1} \Gamma_3(q, q_1)\varphi(q_1)\, dS_{q_1} \tag{5.5}$$
$$- \lambda\int_{S_2} \Gamma_1(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}_2(q) \quad (q \in S_2),$$

where $\Gamma_3(q, q_1) = T_{\nu(q)}\Gamma_2(q, q_1)$. In the second of these equations, it is possible to interchange the order of integration and differentiation, since $q \in S_2$. This approach was proposed in a somewhat different form in [114] and for the case of a plane problem, in [115].

We shall mention one more integral equation which may be obtained on the basis of the identities (1.12) or (1.14):

$$\int_{S_1} \Gamma(q, q_1) T_\nu \mathbf{u}(q_1)\, dS_{q_1} + \int_{S_2} \Gamma_2(q, q_1)\mathbf{u}(q_1) dS_{q_1} = \mathbf{F}_3(q) \quad (q \in S_1), \tag{5.6}$$

$$\mathbf{u}(q_1) - \int_{S_1} \Gamma(q, q_1) T_\nu \mathbf{u}(q_1) dS_{q_1} - \int_{S_2} \Gamma_2(q, q_1)\mathbf{u}(q_1) dS_{q_1} = \mathbf{F}_4(q) \quad (q \in S_2),$$

where

$$\mathbf{F}_3(q) = \mathbf{F}_1(q) - \int_{S_2} \Gamma(q, q_1)\mathbf{F}_2(q_1) dS_{q_1} - \int_{S_1} \Gamma_2(q, q_1)\mathbf{F}_1(q_1) dS_{q_1},$$

and

$$\mathbf{F}_4(q) = - \int_{S_2} \Gamma(q, q_1)\mathbf{F}_2(q_1) dS_{q_1} - \int_{S_1} \Gamma_2(q, q_1)\mathbf{F}_1(q_1) dS_{q_1}.$$

The signs used here correspond to the interior problem.

Equations (5.6) are also singular equations with discontinuous kernels and discontinuous coefficient of the term outside the integral sign. The solvability conditions for equations of the type (5.6) have also not been investigated. However, it is certainly useful to work out effective numerical methods for solving Eqs. (5.2), (5.5), and (5.6)[22]. For example, it is not difficult to realize the method of mechanical quadratures in some form. In order to calculate the singular integrals in Eqs. (5.2), (5.5), and (5.6), we can use the regular representation (3.2) in a modified form. If, however, we carry out polygonalization of the surface, it is possible to use the cubature formulas [94, 95].

It should be noted that a solution of the mixed problems by the method of functional equations practically does not differ from the case of basic problems. The only difference is that the unknown displacements will be specified on one part of the boundary surface (on a discrete set of points), while the stress vector will be specified on another part.

Let us consider in detail the solution of Eqs. (5.2). The presence of an operator of the first kind in this equation renders the problem incorrect, which may be

[22] It should be noted that some mixed problems have already been considered on the basis of Eqs. (5.5) and (5.6) [95].

manifested in the instability of some numerical algorithm, although the mixed boundary value problem remains correct[23].

Suppose that the external stress on the surface S_1, which is yet to be determined, is represented in the form of a series (with unknown coefficients) of a certain complete system of functions, multiplied by a function which takes into account the nature of singularity in stresses (determined in accordance with Sec. 8, Ch. 3, Vol. 1). We then arrive at a set of second boundary value problems. Solving these problems somehow, we can find the values of displacements on the surface S_1 in each case. The question now arises about the determination of the coefficients of the series introduced above, by satisfying the boundary conditions on S_1. Here, we can use different methods: the method of collocations, the method of least squares, etc. The algebraic equations for the coefficients obtained by using a finite part of the series may turn out to be ill-conditioned[24]. Moreover, the condition number increases with increasing order of the system. The incorrectness of the equations of the first kind is manifested in this way. The regularizing algorithms, described in Sec. 16, Ch. 1, Vol. 1, can be used for solving such systems.

Let us consider the additional difficulties that may arise while solving interior problems. As a matter of fact, the load applied to the body in each case, generally speaking, is found to be non-self-balanced, and hence the boundary value problem is unsolvable. In order to eliminate these difficulties, the simplest way is to introduce at some point a force and a moment which balance the external load, and solve the modified problems thus obtained. It follows from the condition of equilibrium of the body as a whole that in the final solution (after summation), the auxiliary terms introduced earlier cancel out.

It should be noted that the method described above is applicable (and even more effective) when the normal component of the displacements and the tangential components of stresses are given on the surface S_1. In this case only one scalar component (the normal component of stresses) is defined on the surface S_1.

In order to obtain stable algorithms for solving mixed problems, let us consider the modified problem by introducing on the surface S_1 the boundary condition

$$\lim_{p \to q} \; [\alpha T_\nu \mathbf{u}(p) + \mathbf{u}(p)] = \mathbf{F}_1(q), \tag{5.7}$$

where $\alpha > 0$ is a sufficiently small number [116]. On the surface S_2, the boundary condition retains its earlier form (5.1).

For the sake of simplicity, we put $\mathbf{F}_2(q) = 0$, and consider in the domain D the displacements satisfying the homogeneous conditions on S_2. The initial boundary value problem may then be considered as a problem in the space of functions satisfying these conditions. In symbolic form, this problem can be written in the form

$$A\mathbf{u} = \mathbf{F}_1. \tag{5.8}$$

Similarly, for the problem (5.7), we get

[23] The proof for the existence and uniqueness of the solution of this problem is given in Sec. 1, Ch. 8.

[24] This leads to a considerable variation in the coefficients of the series upon varying the order of the system.

$$A[\alpha T_\nu \mathbf{u} + \mathbf{u}] = \mathbf{F}_1. \tag{5.9}$$

In order that the solutions of these two problems (for small α) be close in some spaces, it is sufficient[25] that the operator AT_ν in these spaces be bounded.

The solution of the problem (5.9) is also constructed in the same way by starting from the representations of displacements in the form of a single layer potential. We then get the integral equation

$$\alpha[\varphi(q) - \lambda\int_S \Gamma_1(q, q_1)\varphi(q_1)dS_{q_1}]$$

$$+ \int_S \Gamma(q, q_1)\varphi(q_1)dS_{q_1} = \mathbf{F}_1(q) \quad (q \in S_1), \tag{5.10}$$

$$\varphi(q) - \lambda\int_S \Gamma_1(q, q_1)\varphi(q_1)dS_{q_1} = 0 \quad (q \in S_2);$$

the choice of the sign has been made in the same way as for Eq. (2.1).

Since Eqs. (5.10) differ from equations of the second kind only by a fully continuous operator, it is sufficient to show their uniqueness for the purpose of correct solvability.

In order to prove this uniqueness, we proceed from Betti's first formula (see Sec. 4, Ch. 2, Vol. 1)

$$\int_S \mathbf{u}T_\nu \mathbf{u}dS + \int_D E(\mathbf{u}, \mathbf{u})d\Omega = 0.$$

Transforming the first integral with the help of the boundary conditions, we arrive at the equality

$$\frac{1}{\alpha}\int_{S_1} u_i u_i dS + \int_D E(\mathbf{u}, \mathbf{u})d\Omega = 0. \tag{5.11}$$

Since each integral is a non-negative quantity, the displacements are identically equal to zero. Thus, if we assume that the solutions of the integral equations are not unique, we obtain a single layer potential which vanishes in D, thus rendering the density equal to zero.

Naturally, the choice of the parameter α is determined by the magnitude of the error in the solution of Eqs. (5.10).

Let us now consider some examples. First of all, we shall solve the problem directly on the basis of Eqs. (5.2). We consider [118] the axially symmetric contact problem for a deepened punch. Suppose that in a half-space we have a cylindrical cavity

[25] This question may be investigated by introducing the so-called Sobolev-Slobodetskii spaces [117].

of radius a and height H, to whose bottom a smooth punch of the same radius has been applied. We assume the force p on the punch to be given. We assume that the external stress is equal to zero on the remaining surface of the body. The normal component of the stress is given in the form of a series (the z-axis coincides with the axis of the cylinder)

$$\sigma_z\left(\frac{\rho}{a}\right) = \left[1 - \left(\frac{\rho}{a}\right)^2\right]^{-2/3}\left[\sum_{n=0}^{\infty} \alpha_n\left(\frac{\rho}{a}\right)^{2n}\right]. \tag{5.12}$$

Here, the exponent $-2/3$ is determined from Eq. (8.38), Ch. 3, Vol. 1 for $2\alpha = 1.5\pi$.

Calculations were carried out by retaining one, two, and three terms in (5.12). Table 16 gives the values of the coefficients α_0, α_1, and α_2, the dimensionless displacement $w'(w' = waE/p)$ of the punch, and the maximum error in satisfying the boundary condition (in displacements) for two values of H/a.

It is quite significant that the displacement w of the punch is practically independent of the number of terms retained in (5.12).

We now go over to the axially symmetric problem for a circular plate of radius 10 and thickness 2. There are no stresses on the faces S_2, while on the lateral surface S_1,

$$u_r = 0; u_z = z. \tag{5.13}$$

The stresses on the lateral (contact) surface may be represented in the form of the series

$$\sigma_r(z) = \frac{1}{(z^2 - 1)^{0.31}} \sum_{n=0}^{\infty} a_{2n} T_{2n}(z),$$

TABLE 16

H/a	α_0	α_1	α_2	w'	Δ
	1.000	—	—	0.386	0.0380
0.2	1.276	-0.368	—	0.392	0.0032
	1.266	-0.409	0.126	0.383	0.0030
	1.000	—	—	0.303	0.0132
3.5	1.061	-0.082	—	0.305	0.0055

$$\tau_{rz}(z) = \frac{1}{(z^2 - 1)^{0.31}} \sum_{n=0}^{\infty} a_{2n+1} T_{2n+1}(z), \tag{5.14}$$

where $T_n(z)$ are the Chebyshev polynomials. The factor appearing in front of the summation sign takes into consideration the nature of the singularity in accordance with Eq. (8.37), Ch. 3, Vol. 1.

Hence, the solution of the integral equation (5.10) will also be sought in the form of the series

$$\varphi(q) = \sum_{k=0}^{\infty} a_k \varphi_k(z), \tag{5.15}$$

where $\varphi_k(z)$ is the solution of the integral equation of the second basic problem under the conditions

$$T_\nu \mathbf{u}|_{S_2} = 0, \; T_\nu \mathbf{u}|_{S_1} = \mathbf{\Phi}_k(z),$$

$$\mathbf{\Phi}_k(z) = \begin{cases} \dfrac{1}{(z^2 - 1)^{0.31}} [T_k(z), 0] & (k \text{ is even}), \\ \\ \dfrac{1}{(z^2 - 1)^{0.31}} [0, T_k(z)] & (k \text{ is odd}). \end{cases}$$

Then, for any values of the coefficients a_k, the boundary conditions on the surface S_2 are automatically satisfied. Turning to the condition (5.9), we arrive at the equation

$$\alpha \sum_{k=0}^{\infty} a_k \mathbf{\Phi}_k(z) + \sum_{k=0}^{\infty} a_k \mathbf{u}_k(z) = \mathbf{F}_1(z), \tag{5.16}$$

where

$$\mathbf{u}_k(p) = \int_S \Gamma(p, q) \varphi_k(q) dS_q.$$

This equation can be used for determining the coefficients a_k.

The effect of regularization is most marked on the calculated values of the contact stresses. Without regularization, they are not observed to acquire stable values (especially for the tangential components). The reason behind this is that for higher harmonics, the coefficients a_k have large values.

It should, however, be noted that as we move away from the contact surface, the differences in the stresses will quickly disappear.

Let us now consider a special class of contact problems, when a smooth punch acts on the surfaces S_1 of an elastic body in the form of a half-space, while the stresses vanish outside the punch. As mentioned in Sec. 5, Ch. 3, Vol. 1, the solution in this case is reduced to the determination of a harmonic function in the half-

space, such that its normal derivative vanishes at the regions of the boundary surface, where the stresses are given.

We can approach the solution of this problem as follows. Suppose that $p(q)$ is the pressure acting on the contact surface. Then, integrating (5.24), Ch. 3, Vol. 1, we get a representation for displacements in the entire half-space. The expression for the normal displacement w at the boundary acquires an especially simple form:

$$w = \nu_1 \int_{S_1} \frac{p(q)}{r(q', q)} \, dS_q \quad \left(\nu_1 = \frac{1 - \nu^2}{\pi E}\right). \tag{5.17}$$

We then get the following integral equation for $p(q)$:

$$\int_{S_1} \frac{p(q)}{r(q', q)} \, dS_q = f(q), \tag{5.18}$$

where $f(q)$ is a given function defined by the geometry of the punch. For the sake of simplicity, $p(q)$ has been normalized. In symbolic form, Eq. (5.18) can be written as follows:

$$Ap = f. \tag{5.18'}$$

Equation (5.18) is a Fredholm integral equation of the first kind, and is incorrect, as mentioned in Sec. 16, Ch. 1, Vol. 1. In order to obtain a stable solution, we can resort to the usual methods of regularization.

We shall now describe a somewhat simpler method of regularization, which takes into account the property of the kernel of this equation, i.e. the eigenvalues of the operator are positive. We shall prove this property [119].

Let D_R be a domain cut from the half-space by a sphere S_R of radius R, having its centre inside $S_1(R_1 > \operatorname{diam} S_1)$. According to Betti's first formula (4.26), Ch. 2, Vol. 1, we have

$$\int_{D_R} E(\mathbf{u}, \mathbf{u}) d\Omega = \int_{S_1} \mathbf{u} T_\nu \mathbf{u} dS + \int_{S_3} \mathbf{u} T_\nu \mathbf{u} dS \geqslant 0,$$

where S_3 is the surface of a hemisphere. The right-hand side does not contain an integral taken over the plane boundary of a hemisphere lying outside S_1, since the integrand in this case vanishes. For $R \to \infty$, we arrive at the inequality

$$\int_{S_1} \mathbf{u} T_\nu \mathbf{u} dS = \int_{D} E(\mathbf{u}, \mathbf{u}) d\Omega \geqslant 0. \tag{5.19}$$

Thus, we get the required result

$$(Ap, p) = (p, Ap) \geqslant 0. \tag{5.20}$$

The property proved above permits us to directly construct the regularizing operator, thus by-passing the variational formulation for the smoothing functional.

It is well known [120] that the instability of the solution of equations of the first kind is due to a condensation of their eigenvalues at zero, thus rendering the inverse operator infinite. A shift of the spectrum by a positive quantity eliminates this drawback. We perform this shift by going over to the equation

$$\alpha p_\alpha(q') + \int_{S_1} \frac{p_\alpha(q)}{r(q', q)} dS_q = f(q') \quad (\alpha > 0), \tag{5.21}$$

which corresponds to the following modification of the boundary value problem:

$$\begin{gathered} \alpha(T_z\mathbf{u}) + w = f, \tau_{xz} = \tau_{yz} = 0 \quad (q \in S_1), \\ T_\nu \mathbf{u} = 0 \qquad (q \bar{\in} S_1). \end{gathered} \tag{5.22}$$

Naturally, the choice of the parameter α must be compatible with the accuracy of the solution of Eq. (5.21).

Figure 77 shows the solutions for a plane circular punch. The solid line corresponds to the exact solution, the squares correspond to the value $\alpha = 0.05$ upon a subdivision along the radius into 100 equal parts, while the triangles correspond to the value $\alpha = 0$ when a division into 250 equal parts is carried out (the integration over the angle was carried out in the closed form).

It should be noted that in all the cases indicated here, the dependence between the force and the displacement of the punch can be described by a fairly stable quantity even when regularizing algorithms were not used in calculations.

It follows from general considerations (and the examples given above confirm this) that the instant when instability appears in the solutions depends on the algorithm employed and the accuracy of approximation of the operator.

On the basis of the above discussion, we can expect to obtain reliable solutions in

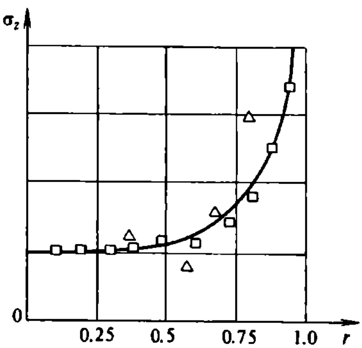

Fig. 77. Contact stresses for a plane circular punch (for different discretizations and different values of parameter α).

a number of cases without resorting to regularizing algorithms[26].

Let us consider a method [122] of constructing a solution of the integral equation (5.17) when the surface S_1 is nearly a circle. We carry out some mapping of the domain S_1 onto a circle S of unit radius. If this mapping is carried out with the help of complex variables z and ζ ($|\zeta| \leqslant 1$), it may be written in the form

$$z = \zeta + \varepsilon\varphi(\zeta), \tag{5.23}$$

where ε is a small parameter.

In terms of the new variables, the integral equation (5.17) can be written in the form

$$\int_S K(\zeta, \zeta')p(\zeta)dS = Kp = f(\zeta), \tag{5.24}$$

where

$$K(\zeta, \zeta') = \frac{I(\zeta')}{\{\zeta - \zeta' + \varepsilon[\varphi(\zeta) - \varphi(\zeta')]\}^{1/2}\{\bar{\zeta} - \bar{\zeta}' + \varepsilon[\overline{\varphi(\zeta)} - \overline{\varphi(\zeta')}]\}^{1/2}}$$

$$= \frac{I(\zeta')}{(\zeta - \zeta')^{1/2}(\bar{\zeta} - \bar{\zeta}')^{1/2}(1 + 2\varepsilon|t|\cos\psi + \varepsilon^2|t|^2)^{1/2}}. \tag{5.25}$$

Here, $t = \dfrac{\varphi(\zeta) - \varphi(\zeta')}{\zeta - \zeta'} = te^{i\psi}$, $I(\zeta)$ is the Jacobian transform, and the same notation has been retained for the quantities $p(z(s))$ and $f(z(s))$.

If the function $\varphi(\zeta)$ is analytical, we get

$$I(\zeta) = 1 + \varepsilon[\varphi'(\zeta) + \overline{\varphi'(\zeta)}] + \varepsilon^2\varphi'(\zeta)\overline{\varphi'(\zeta)}.$$

We transform the kernel of (5.25) by introducing a new function

$$K_0(\zeta, \zeta') = \frac{1}{(\zeta - \zeta')^{1/2}(\bar{\zeta} - \bar{\zeta}')^{1/2}} \tag{5.26}$$

and expanding the remaining terms into a series in Legendre polynomials.

This gives

$$K(\zeta, \zeta') = K_0(\zeta, \zeta')I(\zeta')\left[\sum_{n=0}^{\infty} (-1)^n\varepsilon^n|t|^n P_n(\cos\psi)\right]. \tag{5.27}$$

It should be noted that the kernel $K_0(\zeta, \zeta')$ is the kernel of the integral equation (5.17) for the case when the domain S_1 is a circle. In this case, the equation can be written in a symbolic form as follows:

$$\int_{S_1} K_0(\zeta, \zeta')p(\zeta)dS \equiv K_0p = f. \tag{5.28}$$

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[26] It should be noted that a special type of cubature formulas has been used in [121].

We can represent Eq. (5.24) in the following form:

$$K_0 p = f - (K - K_0)p, \tag{5.29}$$

which is convenient for solving it by the method of successive approximations by proceeding from the relations

$$p_n = p_0 - K_0^{-1}(K - K_0)p_{n-1} = \sum_{n=0}^{\infty} (-1)^k [K_0^{-1}(K - K_0)]^k p_0. \tag{5.30}$$

Here, K_0^{-1} is an operator[27] inverse to K_0, and p_0 is the solution for a circular punch.

In the final stage for determining the contact pressure, it is necessary to go over from the function $p(\zeta)$ to the function $p(z)$.

In Sec. 8, Ch. 1, Vol. 1, we mentioned a class of mixed boundary value problems for Laplace equations for the case of a half-space with the line of demarcation of the boundary conditions along an ellipse. In this case, an effective solution can be constructed in an explicit form. By this we mean that inside the ellipse we are given the function u which is a polynomial of degree n, while outside the ellipse the normal derivative vanishes. The expression for the normal derivative on the elliptical surface element can be written in the following form in this case:

$$\frac{\partial u}{\partial n} = \frac{\Phi_n(x, y)}{\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}}, \tag{5.31}$$

where $\Phi_n(x, y) = \sum\limits_{i,j}^{i+j\leqslant n} c_{ij}x^i y^j$ is a polynomial of the same order, and a and b are the semi-axes of the ellipse.

This result is used for solving the contact problem for the case when the smooth punch is in the form of an ellipse. Indeed, the following dependence holds between the displacements $w(x, y)$ and the contact stresses $p(x, y)$:

$$w(x,y) = \int_S \frac{p(x_1, y_1)}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} dx_1 dy_1. \tag{5.32}$$

The functional equation applicable for the case under consideration is given below:

$$\gamma + \alpha x + \beta y - f_n(x, y)$$
$$= \int_S \frac{\Phi_n(x_1, y_1)dx_1 dy_1}{\sqrt{(x - x_1)^2 + (y-y_1)^2}\sqrt{1-\dfrac{x_1^2}{a^2}-\dfrac{y_1^2}{b^2}}}, \tag{5.33}$$

[27] The expression for this operator is given by (8.11), Ch. 1, Vol. 1.

where $f_n(x, y)$ is the equation for the surface of the punch in the form of a polynomial, γ, α, and β are constants determining the displacement of the punch as a rigid body[28]. This equation permits us to calculate the coefficients of the polynomial $\Phi_n(x, y)$. In the mechanical formulation of the problem, the principal vector **P** of the forces applied to the punch (since there are no shearing stresses, thi; vector consists of just one component P_z) and the point (x_0, y_0) of its application are assumed to be given. Thus, three scalar quantities are given, which is equivalent to three constants α, β, and γ. Conversely, it may be assumed that these constants are given. Then the force characteristics are determined by integration of the stress $p(x, y)$. The expressions for all the integrals appearing in (5.33), in accordance with [123], for the case $n = 2$ are given below:

$$
\begin{aligned}
&\int_S \frac{dx_1 dy_1}{\sqrt{(x-x_1)^2+(y-y_1)^2}\sqrt{1-\dfrac{x_1^2}{a^2}-\dfrac{y_1^2}{b^2}}} \\
&\qquad = \int_S K(x, y, x_1, y_1) dx_1 dy_1 = 2\pi a k' F(k) = d_{00}^{00}, \\
&\int_S K(x, y, x_1, y_1) x_1 dx_1 dy_1 = \frac{2\pi a k'}{k^2}[F(k) - E(k)]x = d_{10}^{10} x, \\
&\int_S K(x, y, x_1, y_1) y_1 dx_1 dy_1 = \frac{2\pi a k'}{k^2}[E(k) - k'^2 F(k)]y = d_{01}^{01} y, \qquad (5.34) \\
&\int_S K(x, y, x_1, y_1) x_1^2 dx_1 dy_1 = \frac{\pi a^3 k'}{k^2}[E(k) - k'^2 F(k)] \\
&\qquad + \frac{\pi a k'}{k^2}[2F(k) - (2 + k^2)E(k)]x^2 \\
&\qquad + \frac{\pi a k'}{k^4}[2E(k) - (2 - k^2)F(k)]y^2 = d_{20}^{00} + d_{20}^{20} x^2 + d_{20}^{02} y^2, \\
&\int_S K(x, y, x_1, y_1) y_1^2 dx_1 dy_1 = \frac{\pi a^3 k'^3}{k^2}[F(k) - E(k)] \\
&\qquad + \frac{\pi a k'^3}{k^4}[2E(k) - (2 - k^2)F(k)]x^2 + \frac{\pi a k'}{k^4}[2k^4 F(k) \\
&\qquad - (2 - 3k^2)E(k)]y^2 = d_{02}^{00} + d_{02}^{20} x^2 + d_{02}^{02} y^2,
\end{aligned}
$$

[28] The integrals in (5.33) can be calculated comparatively easily, when the polynomial is of the second degree (i.e. when the surface of the punch is a paraboloid). In the expression for $f_n(x, y)$, we assume that the linear terms are absent, and they are automatically accounted for with the help of the coefficients α, β, and γ.

$$\int_S K(x, y, x_1, y_1)x_1y_1dx_1dy_1$$

$$= \frac{2\pi ak'}{k^4}[(2 - k^2)E(k) - 2k'^2F(k)] = d_{11}^{11}xy.$$

Hence, $F(k)$ and $E(k)$ are elliptical functions of the first and the second kind, k is the eccentricity of the ellipse, $k' = \sqrt{1 - k^2}$, and the meaning of the notation d_{ji}^{kl} is obvious.

Thus, the solution of the contact problem (for $n \leqslant 2$) at the final stage can be reduced to the following system of equations:

$$\begin{gathered}
d_{00}^{00}c_{00} + d_{20}^{00}c_{20} + d_{02}^{00}c_{02} = b_{00} = -\gamma, \\
d_{20}^{20}c_{20} + d_{02}^{20}c_{02} = b_{20}, \quad d_{20}^{02}c_{20} + d_{02}^{02}c_{02} = b_{02}, \\
d_{10}^{10}c_{10} = b_{10} = -\alpha, \quad d_{10}^{01}c_{01} = b_{01} = -\beta, \\
d_{11}^{11}c_{11} = b_{11}.
\end{gathered} \tag{5.35}$$

Going over to the force P_z and the coordinates (x_0, y_0) of the point of application of this force, we must use the following additional equations:

$$\begin{gathered}
4\pi^2mGa^2k'[c_{00} + (1/3)a^2c_{20} + (1/3)a^2k'^2c_{02}] = -(m - 1)P_z, \\
4/3\pi^2a^4mGk'c_{10} = -(m - 1)x_0P_z, \\
4/3\pi^2a^4mGk'^3c_{01} = -(m - 1)y_0P_z \quad \left(m = \frac{1}{\nu}\right).
\end{gathered} \tag{5.36}$$

For the case when the punch is plane ($b_{20} = b_{02} = b_{11} = 0$), the system (5.35) has a simple solution. The final results can be given in the following form:

$$\begin{gathered}
p(x, y) = -\frac{P_z}{2\pi a^2k'}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2}\left(1 + \frac{3y_0y}{b^2} + \frac{3x_0x}{a^2}\right), \\
\gamma = \frac{(m - 1)P_zF(k)}{2\pi maG}, \quad \alpha = \frac{3(m - 1)x_0P_z[F(k) - E(k)]}{2\pi ma^3k^2G}, \\
\beta = \frac{3(m - 1)y_0P_z[E(k) - k'^2F(k)]}{2\pi ma^3Gk^2k'^2}.
\end{gathered} \tag{5.37}$$

Let us also consider the case of a parabolic punch ($b_{11} = 0$, $b_{20} > 0$, $b_{02} > 0$), when the resultant force is applied at the point $x_0 = y_0 = 0$. The solution in this case also can be found quite easily, since the system (5.35) can be reduced to a system of the third (and practically second) order:

$$\begin{gathered}
d_{00}^{00}c_{00} + d_{20}^{00}c_{20} + d_{02}^{00}c_{02} = -\gamma, \\
d_{20}^{20}c_{20} + d_{02}^{20}c_{02} = b_{20}, \\
d_{20}^{02}c_{20} + d_{02}^{02}c_{02} = b_{02}.
\end{gathered} \tag{5.38}$$

On the basis of the above formulas, it is possible to consider the problem of indentation of a punch whose surface is not bounded *a priori* by a boundary contour, but is rather determined in the course of solution of the problem, depending on the applied force. The general case cannot be considered here[29], since the apparatus used for this purpose is closely linked with the fact that the surface of contact is an ellipse. The only possibility of obtaining a solution with the help of this method is directly related to the question as to which parameters make the solution such that the polynomial $\Phi_n(x, y)$ may be represented as a product of the function $(1 - x^2/a^2 - y^2/b^2)$ and a certain factor. The problem formulated here has several solutions. We shall consider one of these, discovered by Hertz, in which the additional polynomial is a constant. In this case, the punch is a paraboloid (b_{20}, $b_{02} > 0$, $b_{11} = 0$). Substituting the values $c_{00} = c$, $c_{20} = -c/a^2$, $c_{02} = -c/b^2$ into (5.35), we find that the system (5.38) is compatible for definite values of eccentricity. The following transcendental equation is valid for the eccentricity of the elliptical surface element:

$$\frac{k^2}{(1 - k^2)F(k)/E(k) - 1} = \frac{b_{20} + b_{02}}{b_{02}}. \tag{5.39}$$

The lengths of the semi-axes are expressed in terms of the forces P_z as follows:

$$a = \sqrt{\frac{6(m - 1)P_z[F(k) - E(k)]}{\pi m G k^2 b_{20}}}, \quad b = \frac{3P_z(m - 1)F(k)}{4\pi a m G}. \tag{5.40}$$

The expression for the contact pressure has the form

$$p(x, y) = -\frac{3P_z}{2\pi a^2 k'^2}\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}. \tag{5.41}$$

It should be noted that since the eccentricity is independent of the magnitude of the force, the entire set of boundary contours (for different forces) forms an aggregate of confocal ellipses. The results obtained above show that, as in the case of the plane problem, the stress at the boundary points is either infinite or equal to zero.

We shall now consider the problem of compression of two elastic bodies, when the contact area is small in comparison to their dimensions, and we can approximately go over to the problem of compression of two half-planes ($z > 0$, $z < 0$). Suppose that the equations of areas in contact are

$$z_1 = f_1(x, y), \quad z_2 = f_2(x, y).$$

We shall assume that the points having the same coordinates x and y come into contact upon compression[30]. We can then write the following conditions for the

[29] Specific arguments concerning this case have been described in Sec. 1, Ch. 8.

[30] This assumption is equivalent to neglecting the displacements in the tangential plane.

displacements $w_1(x, y)$ and $w_2(x, y)$:

$$w_1 + w_2 = \delta - f_1(x,y) - f_2(x,y) = f(x,y), \tag{5.42}$$

where δ characterizes the distance by which the bodies come closer.

Under the restrictions imposed above, it is expedient, for establishing a relation between the contact pressure $p(x, y)$ and displacements, to apply the relation (5.17) to each of the half-spaces:

$$w_1(x, y) = \frac{1 - \nu_1}{2\pi G_1} \int\limits_S \frac{p(x_1, y_1)}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} dx_1 dy_1 \tag{5.43}$$

$$= \frac{\theta_1}{2\pi} \int\limits_S \frac{p(x_1, y_1)}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} dx_1 dy_1,$$

$$w_2(x, y) = \frac{\theta_2}{2\pi} \int\limits_S \frac{p(x_1, y_1)}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} dx_1 dy_1. \tag{5.44}$$

Then, we get the following relation from Eq. (5.42):

$$\frac{\theta_1 + \theta_2}{2\pi} \int\limits_S \frac{p(x_1, y_1)}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} dx_1 dy_1 = f(x, y). \tag{5.45}$$

Thus, we again arrive at the integral equation for a rigid punch indented in the half-space, which enables us to use the previous results.

The solution proposed above for an elliptical punch is considerably simplified for the case of a circular punch. However, in this case, we have an expression for Green's function (see Sec. 8, Ch. 1, Vol. 1), and hence the solution may be obtained directly, without imposing the above-mentioned restrictions on the surface of the punch.

Let us consider a particular axially symmetric problem [124]. Suppose an elastic sphere of radius R is supported by a smooth spherical base whose shape is given by the following equation:

$$r = R[1 + \rho(\theta)] \, (\rho(0) = 0, \quad 0 \leqslant \theta < \gamma, \quad \rho(\theta) \ll 1). \tag{5.46}$$

The remaining part of the surface ($\gamma \leqslant \theta \leqslant \pi$) is subjected to the normal load $N(\theta)$. In order to solve this problem, we use the formulas (1.30), Ch. 4, expressing the relation between the normal stresses applied to the sphere and the displacement of its surface.

For the unknown function, we take the required contact stress $\sigma_r(\theta)$. Note the necessary condition

$$2\pi R^2 \int_0^{\gamma} \sigma_r(\alpha)\sin\alpha \cos\alpha \, d\alpha = -P_z \quad (P_z > 0), \tag{5.47}$$

where P_z is the resultant of external forces.

From the condition (5.46), we obtain the boundary condition in displacements:

$$u_r = R[-a\cos\theta + \rho(\theta)], \tag{5.48}$$

where a is the required distance by which the base approaches the centre of the sphere.

Then from the first formula of (1.27), Ch. 4, we get the following integral equation for the contact pressure $\sigma_r(\theta)$:

$$\frac{R}{2\pi\mu} \int_0^{\gamma} \sigma_r(\alpha) H_r(1, \theta, \alpha)\sin\alpha \, d\alpha = U(\theta) \quad (0 < \theta < \gamma), \tag{5.49}$$

where

$$U(\theta) = R\left[\rho(\theta) - a\cos\theta - \frac{1}{2\pi G} \int_{\gamma}^{\pi} N(\alpha) H_r(1, \theta, \alpha)\sin\alpha \, d\alpha\right].$$

Carrying out a change of variables

$$\tan\frac{\theta}{2} = \varepsilon x, \quad \tan\frac{\alpha}{2} = \varepsilon t, \quad \varepsilon = \tan\frac{\gamma}{2}$$

and introducing the notation

$$\sigma^0(x) = \sigma_r(2\arctan \varepsilon x),$$

$$U^0(x) = U(2\arctan \varepsilon x),$$

$$U^0(y) \equiv U^0(y, x, t) = U(y, 2\arctan \varepsilon x, 2\arctan \varepsilon t),$$

$$q(x) = \frac{4\varepsilon^2 \sigma^0(x)}{(1 + \varepsilon^2 x^2)^{3/2}\mu},$$

$$w(x) = \frac{2U^0(x)}{R(1 + \varepsilon^2 x^2)^{1/2}}, \quad \nu_1 = \frac{1 - \nu}{2\pi},$$

$$S(x, t) = \frac{t}{\sqrt{(1 + \varepsilon^2 x^2)(1 + \varepsilon^2 t^2)}} \left\{\frac{1}{2} \cdot \frac{1 - 2\nu}{1 + \nu} - 2(1 - \nu)\left[1 + \frac{(1 - \varepsilon^2 x^2)(1 - \varepsilon^2 t^2)}{(1 + \varepsilon^2 x^2)(1 + \varepsilon^2 t^2)}\right] + \frac{1}{\pi} \operatorname{Re} \int_0^1 \left(\frac{P + Q}{y^{1 + n_1}} + \frac{1}{y^2}\right) U^0(y) dy\right\},$$

we can transform Eq. (5.49) as follows:

$$\int_0^1 q(t)\left[\frac{4t}{x+t}K\left(\frac{2\sqrt{xt}}{x+t}\right)+\frac{\varepsilon}{\nu_1}S(x,t)\right]dt=\frac{\varepsilon}{\nu_1}w(x) \tag{5.49'}$$
$$(0<x<1).$$

In turn, Eq. (5.47) assumes the form

$$\int_0^1 q(t)\frac{(1-\varepsilon^2t^2)t}{(1+\varepsilon^2t^2)^{3/2}}dt=-\frac{P_z}{2\pi R^2\mu}. \tag{5.47'}$$

It should be noted that the axially symmetric contact problem for a half-space may be reduced to the integral equation

$$\int_0^1 q(t)\frac{4t}{x+t}K\left(\frac{2\sqrt{xt}}{x+t}\right)dt=f(x)\ (0<x<1). \tag{5.50}$$

Following [125], we can represent the solution (8.11), Ch. 1, Vol. 1, of this equation in the form

$$q(x)=\frac{c}{\sqrt{1-x^2}}-\frac{\sqrt{1-x^2}}{\pi^2}$$
$$\times\int_0^{\pi/2}d\psi\int_0^{\pi/2}\Delta f(\sqrt{1-(1-x^2)\sin^2\varphi}\,\sin\alpha)\sin\psi\sin\alpha\,d\alpha, \tag{5.51}$$

where

$$\Delta f(t)=\frac{1}{t}f'(t)+f''(t),\quad c=\frac{1}{\pi^2}\left[f(0)+\int_0^1\frac{f'(t)dt}{\sqrt{1-t^2}}\right].$$

We can show that the solution (5.51) may be represented in the form

$$q(x)=\frac{c}{\sqrt{1-x^2}}-\frac{1}{\pi^2}\int_0^1\Delta f(t)L(x,t)dt, \tag{5.51'}$$

where

$$L(x,t)=\frac{t}{x+t}K\left(\frac{2\sqrt{xt}}{x+t}\right)-F\left(\frac{x}{t},t\right).$$

The integral on the right-hand side of (5.51′) is considered as an operator. We act from the left on (5.49) with the help of this operator. This leads to the equivalent equation

$$q(x) - \frac{\varepsilon}{\nu_1\pi^2}\int_0^1 q(t)B(x,t)dt$$
$$= \frac{c}{\sqrt{1-x^2}} - \frac{\varepsilon}{\nu_1\pi^2}\int_0^1 \Delta w(t)L(x,t)dt \qquad (0 < x < 1), \tag{5.52}$$

where

$$B(x,t) = \int_0^1 \left[\frac{1}{z} S_z'(z,t) + S_{zz}(z,t)\right] L(x,z)dz.$$

Thus, we have obtained an integral equation of the second kind for $q(x)$. In order to solve this equation, we must apply the condition (5.47′).

Section 6
Problems for Bodies with Cuts

Suppose that in the space occupied by an elastic body, we have an unclosed smooth surface S_1, bounded by a smooth contour L whose sides are denoted by S^+ and S^- (in accordance with the positive direction of the outward normal). We assume that the stresses on the sides S^+ and S^- are given[31] by

$$\lim_{p \to q^+} T_\nu \mathbf{u}(p) = \mathbf{F}^+(q^+), \tag{6.1}$$

$$\lim_{p \to q^-} T_\nu \mathbf{u}(p) = \mathbf{F}^-(q^-). \tag{6.2}$$

We put the stresses at infinity equal to zero (if they are not equal to zero, we should go over to a homogeneous condition by superimposing the trivial solution). The restrictions which must be imposed on the functions $\mathbf{F}^+(q^+)$ and $\mathbf{F}^-(q^-)$ will follow from the condition of applicability of the apparatus employed.

The problem posed here belongs to the class of degenerate problems which are not covered by the theory described in Secs. 1-3. This is due not only to the presence of an edge, but mainly on account of the fact that the problem I^+ introduced here (for constructing the theory of the problem II^- whose special case is the problem formulated above) turns out to be a problem for a domain degenerating into a surface, which is devoid of any meaning.

Of course, we may approach the problem formulated above as some limiting solution, when we have a set of surfaces S_j, each covering the other $(D_j^+ \supset D_{j+1}^+)$ and tending in the limit to the two-sided surface S. Then, specifying on them the

[31] Specifying the conditions in a different form is less interesting from a practical point of view.

boundary conditions obtained by a sufficiently smooth extension of the surface S inside the domain, we can arrive at the second exterior problem for the domains D_j^-. Each of these problems is no longer degenerate, and they may be solved by some method. In the limit ($j \to \infty$), we get a solution which should be treated as the solution for a space with a cut. If these solutions are constructed on the basis of the potential theory (Eq. (2.3)), the convergence of successive approximations will deteriorate with increasing j and the corresponding series in the limit will diverge. We shall explain this fact somewhat later. Similar difficulties arise if we take Eq. (2.5) as the basis for the solution. True, in this case the right-hand side tends to zero in the limit. In spite of this, it is possible [126, 127] to draw reliable inferences on the concentration of stresses in the immediate neighbourhood of the edge of the cut from an analysis of the solutions for comparatively thin cavities.

We continue our analysis of the problem, having first simplified its formulation. If the functions $\mathbf{F}^+(q)$ and $\mathbf{F}^-(q)$ are different, we form a single layer potential $\mathbf{V}(p, \mathbf{f})$, having a density equal to $\mathbf{f} = 0.25(\mathbf{F}^+ - \mathbf{F}^-)$. Then, from the formulas (1.23) (assuming that they are applicable), we find that the displacements $\mathbf{u}' = \mathbf{u} - \mathbf{V}(p, \mathbf{f})$ will satisfy the boundary conditions which are identical as we approach the boundary from either side. As a matter of fact,

$$\begin{aligned} T_\nu^+\mathbf{u}' &= T_\nu^+\mathbf{u} - T_\nu^+\mathbf{V} = \mathbf{F}^+ - 0.5(\mathbf{F}^+ - \mathbf{F}^-) \\ &= 0.5(\mathbf{F}^+ + \mathbf{F}^-) = T_\nu^-\mathbf{u} - T_\nu^-\mathbf{V} = \mathbf{F}^- + 0.5(\mathbf{F}^+ - \mathbf{F}^-). \end{aligned} \tag{6.3}$$

Henceforth, we shall omit the prime on the displacements and operate with the conditions (6.1) and (6.2), starting with the assumption that the functions $\mathbf{F}^+(q)$ and $\mathbf{F}^-(q)$ are identical, which enables us to omit their superscripts.

We introduce the double layer potential $\mathbf{W}(p, \varphi)$ with a density equal to half the displacement jump at the cut. Then, the difference $\mathbf{u}' = \mathbf{u} - \mathbf{W}(p, \varphi)$ will represent the displacements which are continuous together with the stress vector in the vicinity of the cut as well. We then apply Stahl's generalized theorem to the theory of elasticity[32] [129] and find that the displacement $\mathbf{u}'$ is identically equal to zero. In other words, the required displacement $\mathbf{u}$ can be represented in the form of a double layer potential $\mathbf{W}(p, \varphi)$.

The above result allows us to explain the reason behind the divergence of the method of successive approximations with increasing values of the index j. Indeed, for the limit transition considered above, the single layer potential is found to be specified on the two-sided surface. On the other hand, we have just established that the solution can be represented in the form of a double layer potential, specified on a one-sided surface. From an identity of solutions, it will follow that such a transition from one potential to the other is possible only when the density of the single layer potential becomes infinite.

We now require that one of the conditions (6.1) or (6.2) be satisfied. This leads to the basic functional equation

$$\lim_{p \to q} T_\nu \mathbf{W}(p, \varphi) = \lim_{p \to q} T_\nu \int_S \mathbf{\Gamma}_2(p, q')\varphi(q')dS_{q'} = \mathbf{F}(q). \tag{6.4}$$

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[32] Stahl's theorem, first proved for harmonic functions (see [128]), states that two harmonic functions, having on some part of the surface identical values for the functions themselves, as well as for their normal derivatives, are identical.

We now go over to a description of the computational systems for solving the functional equation (6.4). Naturally, we shall proceed from some discretization of the surface S into small domains S_j ($j = 1, 2, \ldots, N$), and a piecewise-smooth representation of the function $\varphi(q)$, referred to the central points q_i of the domains, which are called the reference points, as before. We shall require that Eq. (6.4) be satisfied only at the reference points.

The difficulty with the direct realization of Eq. (6.4) is that it cannot be represented in the form of an integral equation and hence, the procedure of the approximate solution cannot be reduced to cubatures. We shall describe some of the methods for constructing approximate algorithms.

We choose a reference point, say, q_j and construct at this point a discrete analogue of Eq. (6.4), starting from the conditionally specified values $\varphi(q_j)$ at all the reference points $q_j (\varphi(q_j) = \varphi_j)$. It must be remarked that the terms of the integral sums corresponding to the domains S_j ($j \neq j_0$) can be easily calculated. Here, we can interchange the order of integration and differentiation in the expression for stresses. As a result, we get the integral

$$\int\limits_{S-S_{j0}} T_\nu \Gamma_2(q_{j_0}, q)\varphi(q) dS_q. \tag{6.5}$$

In order to preserve the accuracy, it is necessary to introduce additional discretization of the surface in those domains S_j, which are situated in the immediate neighbourhood of the point j_0. However, the density in these regions should be determined from the initially introduced values of φ_j by some sort of interpolation.

Applying some cubature formula to the integral (6.5), we get a sum of the form

$$\sum_{j=1}^{N} \alpha'_{j_0, j}\varphi_j. \tag{6.6}$$

Here, generally speaking, term with φ_{j_0} may also appear if a secondary discretization was introduced, since in that case the value of φ_{j_0} must be used while interpolating the density in the additional domains adjoining the point j_0. Apparently, $\alpha'_{j_0, j}$ is a third-order matrix.

There are a number of methods for determining the term corresponding to the domains S_{j_0} [86, 130, 131]. For example, it has been proposed in [131] to introduce points in the body, lying in the immediate neighbourhood of the point q_{j_0} (it is most convenient to locate them on the normal to the surface). We shall denote these points by $p^l_{j_0}$, the parameter l characterizing the distance to the point q_{j_0}. At these points, we should calculate the components of stresses (starting from the density, given only on S_{j_0}), and then carry out extrapolation to the point q_{j_0}[33]. The values of the stresses thus obtained should then be additionally introduced into the sum (6.5).

[33] A certain justification for this procedure follows from the properties of differentiability of the double layer potential (see Sec. 1).

TABLE 17

n	0.4	0.2	0.15	0.1	0.05
30	1.78139	1.66073	1.63240	1.60930	0.51301
60	1.78159	1.66085	1.63255	1.61014	1.57513
120	1.78159	1.66093	1.63260	1.61023	1.59591
180	1.78159	1.66093	1.63260	1.61023	1.59615

As a result, we get a discrete analogue of Eq. (6.4) at the point φ_{j_0}:

$$\sum_{j=1}^{N} \alpha_{j_0,j}\varphi_j = \mathbf{F}(q_{j_0}). \tag{6.7}$$

In order to preserve the accuracy while calculating stresses for small values of l (they must be present in order that there is no loss in accuracy upon extrapolation as well), it is necessary to carry out a secondary discretization of the domain S_{j_0}. If we use the interpolation of the function $\varphi(q)$, the sum (6.7) will have not only the coefficient α'_{j_0,j_0} which will be different from the coefficient α_{j_0,j_0}, but also several coefficients corresponding to the domain S_j in the immediate neighbourhood of the point j_0.

Naturally, in order to ensure that the stresses are determined accurately at the points $p^l_{j_0}$, and to carry out extrapolation, meticulous calculations are required. Table 17 gives the results of calculations for a model problem. A square surface element was taken, and a vector function of constant (unit) magnitude, directed along the normal to the surface element, was specified on this surface. A double layer potential with a density equal to this value was constructed, and the component σ_z was calculated at different points on the normal to the centre of the square (it was assumed that the xOy plane lies in the plane of the square surface element). While calculating the stresses, a secondary discretization of the region into n^2 equal squares was carried out.

Thus, we find that by a suitable choice of the number n, we can obtain stable values of stresses at points lying quite close to the surface.

Table 18 gives the values of stresses (and practically the values of the coefficient α_{j_0,j_0}) obtained by extrapolation for different points used in calculations.

TABLE 18

n \ l	0.4; 0.3; 0.2	0.4; 0.3; 0.2; 0.1	0.4; 0.3; 0.2; 0.1; 0.05
30	1.55187	1.59233	—
60	1.55195	1.59315	1.54165
120	1.55200	1.59388	1.59138
180	—	—	1.59205

It follows from the data given in the table that the extrapolation process is stable, and its accuracy can be controlled.

We shall mention a method for determining the coefficients α_{j_0, j_0}, which requires a preliminary polygonization of the surface. If, as before, we assume that the density is constant within each domain S_j (which is now a polygon), we can show that the displacements and stresses are expressed in an explicit form, which also leads to a discrete analogue of Eq. (6.4) at some reference point.

Thus, by sorting out all the reference points, we arrive at a system of algebraic equations which are a discrete analogue of the entire Eq. (6.4):

$$\sum_{k=1}^{N} \alpha_{jk}\varphi_k = \mathbf{F}(q_j) \quad (j = 1, 2, \ldots, N). \tag{6.8}$$

Let us briefly mention a method of increasing the accuracy without resorting to a fine discretization of the surface in the vicinity of its edge[34].

For this purpose, it is necessary to introduce, in the representation for the function φ, a factor (in local coordinates in the plane perpendicular to the edge of the cut) determined by a known asymptotics for displacements in the vicinity of a corner point (see Sec. 8, Ch. 3, Vol. 1).

Section 7
Compound Bodies. Piecewise Homogeneous Bodies

Let us consider the problems in the theory of elasticity for compound bodies. Suppose that a body contains surfaces (closed or unclosed) on which the displacement vector is discontinuous. Such discontinuities arise during phase transitions or when the body is "composed" of specially prepared parts in such a way that the cavities in contact do not have the same dimensions. In the latter case, we have to make some assumptions regarding the interaction of bodies on the conjugate surfaces in the course of joint deformation. According to the simplest assumption (from the computational point of view), a complete coupling takes place. It can then be assumed that the discontinuity in the displacement vector remains the same. Generally speaking, it can be assumed that there is also a slippage. In this case we must specify the value of the friction coefficient. If there is no friction, we must assume that the tangential components of stress are equal to zero and the jump in the normal component of displacements is known. The problem becomes extremely complicated if we assume that during deformation the compound body gets disconnected in certain regions (which are to be determined). Naturally, we proceed from the condition of complete coupling in the case of a phase transition. In all cases, it must be assumed that the stress vector does not undergo a discontinuity.

We shall solve the problem formulated above for the case when the coupling is assumed to be complete. Suppose that a domain D_0 is bounded from the outside by

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[34] Otherwise, it would be necessary, since the derivative of the displacements (in the direction normal to the edge of the cut) increases indefinitely.

the surface S_0 and from the inside, by the surface S, and that the domain D_1 is bounded from the outside by the surface S and from the inside, by the surface S_1. The following conditions are satisfied on the surface S:

$$\mathbf{u}_1(q) - \mathbf{u}_0(q) = \mathbf{F}(q), \tag{7.1}$$

$$T_n\mathbf{u}_1(q) = T_n\mathbf{u}_0(q). \tag{7.2}$$

Here, $\mathbf{F}(q)$ is the jump in the displacement vector (stretching) (see similar conditions (6.2) and (6.3), Ch. 5, for the plane problem).

Using the properties (1.21) and (1.24) of the double layer potential $\mathbf{W}(p)$, we can directly go over to an auxiliary problem. In this case, we must proceed from the potential density which is equal to $(1/2)\mathbf{F}(q)$. We can then determine the displacements in the domain D_0 by the relation

$$\mathbf{u}(p) = \mathbf{u}_0(p) - \mathbf{W}\left(p, \ \frac{1}{2}\mathbf{F}(q)\right) \tag{7.3}$$

and the displacements in the domain D_1, by the relation

$$\mathbf{u}(p) = \mathbf{u}_1(p) - \mathbf{W}\left(p, \ \frac{1}{2}\mathbf{F}(q)\right). \tag{7.4}$$

This gives the displacements satisfying Lamé's equations over the entire domain $D = D_0 \cup D_1$. New boundary conditions are imposed in this case.

Obviously, the method described here can be automatically extended to the case when there are several inclusions, and even when some of these inclusions themselves are compound bodies.

The case when the contact boundary is on the outer surface deserves special consideration. In the simplest case (shown in Fig. 78), one domain is bounded by the surface $S_1 \cup S$, while the other is bounded by the surface $S_2 \cup S$ (S is the contact surface). Naturally, the double layer potential

$$\mathbf{W}(p) = \frac{1}{2}\int_S \Gamma_2(p, q)\mathbf{F}(q)\, dS_q \tag{7.5}$$

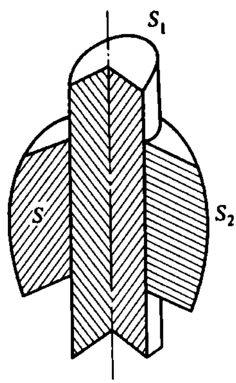

Fig. 78. A compound body.

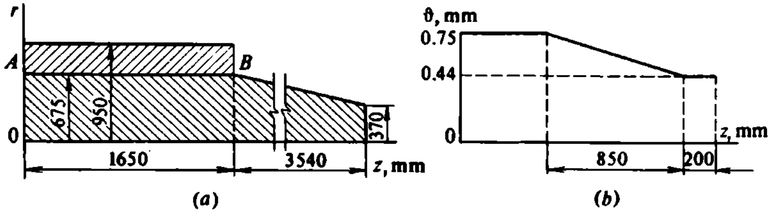

Fig. 79. A bandaged roller.

also leads to the problem for the total domain bounded by the surface $S_1 \cup S_2$. Of course, here we encounter difficulties in determining the additional boundary conditions (in stresses) in the neighbourhood of the contour bounding the unclosed surface S. It should be noted that, generally speaking, these additional conditions suffer a discontinuity as we go over from the surface S_1 to the surface S_2. The first difficulty can be overcome by going over to the double layer potential extended to a surface obtained by supplementing S by a sufficiently narrow strip. On this strip, we specify the density obtained by a continuous extension of the jump function for stretching. On the other edge the density vanishes. The existence of the discontinuity in the boundary conditions for stresses necessitates, as usual, the breaking up of the surface into smaller parts in the vicinity of the boundary contour while solving the problem for the total domain.

The stresses generated by the double layer potential in the vicinity of the surface on which the density is given may be determined by extrapolation, see Sec. 6 for details [131].

Let us consider some examples. The axially symmetric problem associated with the shrink fitting of a hollow cylinder on a solid cylinder has been considered in [132]. This problem emerged while calculating the assembling stresses in a bandaged roller of a rolling mill (the roller is made of two parts—the roller axis and the bandage) (Fig. 79). By forming the roller axis to a suitable shape (by choosing the form of its outer surface) we can attain such stresses on the contact surface, which ensure that the roller works as one piece (without any slippage of the bandage) for comparatively low stresses in the bandage.

Such problems were treated as plane problems, and hence the tangential component of the stress on the contact surface was equal to zero.

Let us determine the magnitude of the jump in displacements. In the case of the normal component, this jump is clearly equal to the difference in the radii. In order to determine the jump in the tangential component of displacements (directed along the generatrix), we must analyze the assembly technique. Suppose that the bandage, upon cooling, is in contact with the roller axis over the entire surface at the same time (assuming that the heating of the roller axis by the hot bandage is not taken into account). Then, assuming further that complete coupling takes place, we may consider that the jump in displacements is a linear function of the axial coordinate, determined by the temperature difference at the moment of contact. Since there is a

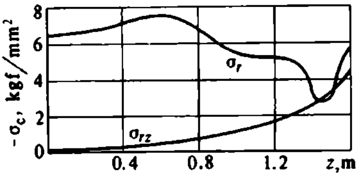

Fig. 80. Diagram of contact stresses.

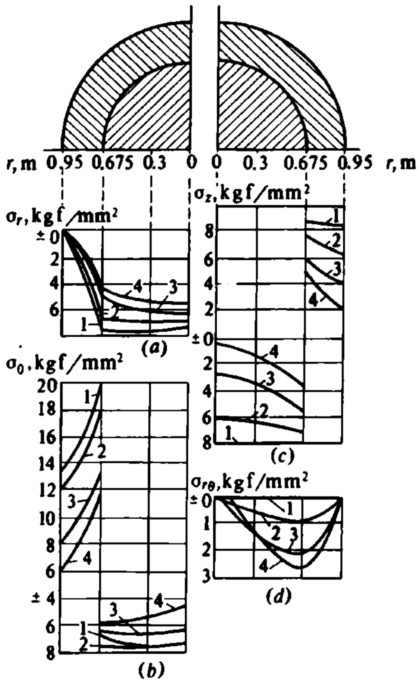

Fig. 81. Stress components in a bandage and the roller axis for different cross sections.

symmetry plane in the problem under consideration, this jump becomes equal to zero on this plane. It follows from the above discussion that, in the general case, the determination of the jump in the tangential component of the displacements (when the materials do not come in contact at the same time, or when there is a slippage with some friction) is quite a complicated problem.

Figure 80 shows the results of calculations of the normal and tangential components of the contact stresses, while Fig. 81 gives the values of the component of

the stress tensor in several cross sections of the bandage and in the roller axis. While making these calculations, it was assumed (to take into account the admissible slippage) that the jump in the tangential component is equal to half the value obtained for the case of complete coupling.

Let us consider a particular case which is of great practical interest. Suppose that the surface S is a part of a cylindrical surface,while the stretching (displacement jump) is constant and directed along the radius. Then the additional terms may be determined from the solution of Lamé's problem for plane strain. For the internal domain, the additional term is given by

$$\sigma_r = \sigma_\theta = -\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\frac{\delta}{R}, \quad \delta_z = -\frac{2\mu\lambda}{\lambda+2\mu}\,\frac{\delta}{R}, \tag{7.6}$$

while for the outer domain it has the form

$$\sigma_r = -\sigma_\theta = -\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\frac{\delta R}{r^2}, \quad \sigma_z = 0. \tag{7.7}$$

Here, δ is the stretching, and R is the radius of the cylindrical surface.

Let us now consider the problem of compound bodies when the inclusions are made of other materials. We analyze a more general case where an elastic body is made up of bodies with different elastic constants, i.e. the case of piecewise homogeneous bodies. By way of an example, let us consider a general problem.

Suppose that we have a set of surfaces $S_1, S_2, \ldots, S_m$, situated outside one another and a surface S_0, such that all the remaining surfaces are situated either inside or outside this surface (the surface S_0 may not exist). We assume that the domain D_0, enclosed between the surfaces $S_0, S_1, \ldots, S_m$, is filled by a medium with constants λ_0 and μ_0 and the domains D_j^+ are filled by media with constants λ_j and μ_j $(j = 1, 2, \ldots, m)$. Thus, the surfaces S_j $(j \neq 0)$ are the surfaces of contact for the media. Some sort of boundary condition is given on the surface S_0. When the surface S_0 is missing (i.e. when the domain D_0 extends to infinity), the behaviour of the solution at infinity must be specified.

A more general formulation of problems for a piecewise homogeneous medium admits cases when the domains D_j^+ themselves are filled by an inhomogeneous medium, as well as cases when the surfaces of demarcation of media extend to the outer surface. A special case of such problems is the problem described in Sec. 5 about the compression of two half-spaces.

It is most natural to reduce the problems about the piecewise homogeneous bodies to a set of problems for each of the domains (filled by a homogeneous medium), introducing auxiliary functions at each surface of contact. If, for example, we specify the external stresses, solve (in the general form) the set of boundary value problems obtained, and determine the displacements on the contact surfaces, we get equations for the stresses introduced by equating these displacements.

It is convenient to realize this approach with the help of potential theory if we start from the integral equations obtained on the basis of the formulas (1.12) and (1.14). We then get a system of integral equations with regular and singular kernels. The unknown quantities will be the displacements and stresses on the contact surfaces, as well as the functions specified at the surface S_0.

Further, we start from the condition whose realization is the easiest. We assume that coupling takes place at the contact surfaces. In a more general form, these conditions can be written as follows:

$$\mathbf{u}_j(q) - \mathbf{u}_0(q) = \mathbf{F}_{1j}(q), \tag{7.8}$$

$$\overset{\lambda_j\mu_j}{T_\nu}\mathbf{u}_j(q) - \overset{\lambda_0\mu_0}{T_\nu}\mathbf{u}_0(q) = \mathbf{F}_{2j}(q)\ (q \in S_j, j \neq 0). \tag{7.9}$$

The meaning of the notation introduced in (7.9) for the operator T is obvious. The functions $\mathbf{F}_{1j}$ and $\mathbf{F}_{2j}$ must be specified. The physical meaning of the functions $\mathbf{F}_{1j}$ is obvious: they define the amount of "tightness" with which one body (inclusion) has been inserted into the other.

It should be mentioned that if all the surfaces of demarcation of media are closed, singular integral equations (with kernels of quite complicated structures) have been obtained [133]. It has been proved that these equations are always solvable (if the domain D_0 is finite, the condition of equilibrium for the body as a whole must be satisfied).

Let us consider in detail the special case in which the Poisson ratios are the same for all media (i.e. the equalities $\lambda_j/\lambda_0 = \mu_j/\mu_0 = k_j$ hold). We can now obtain, without any difficulty, singular integral equations which have practically the same structure as the equations for a homogeneous medium. For this purpose, we rewrite the condition (7.9) in the following form:

$$k_j \overset{\lambda_0\mu_0}{T_\nu}\mathbf{u}_j(q) - \overset{\lambda_0\mu_0}{T_\nu}\mathbf{u}_0(q) = \mathbf{F}_{2j}(q) \tag{7.10}$$

Further, for the sake of simplicity, we shall put $\mathbf{F}_{1j}(q) = 0$, and for the sake of definiteness, we assume that the stresses $\mathbf{F}_0(q)$ are given on the surface S_0. We introduce, everywhere in the total domain D $(D_0 \cup D_1^+ \cup \ldots \cup D_m^+)$, a single layer potential $\mathbf{V}(p, \varphi)$, whose density is distributed over the surfaces S_j.

The symbol φ denotes the set of functions φ_j, which are defined on the surfaces with the corresponding index. This potential is constructed with the help of constants λ_0 and μ_0.

Then, for the density φ, we get the following singular integral equations (to be more precise, a system of equations):

$$\begin{gathered}\varphi_0(q') + \sum_{j=0}^{m}\int_{S_j}\Gamma_1(q', q)\varphi_j(q)dS_q = \mathbf{F}_0(q') \quad (q' \in S_0),\\ \varphi_i(q') + \alpha_i\sum_{j=0}^{m}\int_{S_j}\Gamma_1(q', q)\varphi_j(q)dS_q = \frac{1}{1+k_i}\mathbf{F}_{2i}(q')\\ \left(q' \in S_i;\ \alpha_i = \frac{1-k_i}{1+k_i}\right).\end{gathered} \tag{7.11}$$

These equations differ from Eqs. (2.3) (for the case of a domain bounded by several surfaces) in that some of the integral terms contain factors α_i, which are

modulo less than unity. Specifying the displacements on the surface S_0 will lead to apparent variations. In view of the identity of the structure of the kernels, Fredholm's alternatives can be automatically applied, starting from the results of Sec. 2.

If the surface S_0 is missing and all inclusions are made of the same material, ($\alpha_1 = \alpha_2 = \ldots = \alpha_m = \text{const}$), the convergence of the method of successive approximations follows from the spectral properties of Eqs. (2.3) described in Sec. 2 (we should put the parameter $\lambda = \alpha$ in Eq. (7.9)).

Chapter Eight

Variational and Difference Methods in the Theory of Elasticity

Section 1
Variational Methods in the Theory of Elasticity

We continue our investigation of three-dimensional problems in the theory of elasticity, using a different, variational, approach. We start from the equilibrium equations (4.4″), Ch. 2, Vol. 1:

$$(\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \mathbf{P}, \tag{1.1}$$

which can be written in a symbolic form as follows:

$$A\mathbf{u} = \mathbf{P}. \tag{1.1$'$}$$

We shall directly proceed with the investigation of the mixed problem for a finite domain D, when the boundary S consists of two parts, S_1 and S_2, and the following conditions are satisfied:

$$\mathbf{u}\,|_{S_1} = 0, \quad T_\nu \mathbf{u}\,|_{S_2} = 0. \tag{1.2}$$

Thus, we have an operator (differential) equation with homogeneous boundary conditions. We shall show that this problem (including its limiting case when one of the surfaces, S_1 or S_2, is missing) can be reduced to the problem of minimization of a certain functional under appropriate boundary conditions

$$F(\mathbf{u}) = \int_D [W(\mathbf{u}) - \mathbf{u}\cdot\mathbf{P}]\,d\Omega, \tag{1.3}$$

where $W(\mathbf{u})$ is the potential energy of deformations (see Sec. 3, Ch. 2, Vol. 1).

We introduce a Hilbert space $L_2(D)$ of vector functions $\mathbf{u}(p)$, having a bounded integral $\int_D |\mathbf{u}(p)|^2\,d\Omega < \infty$. The scalar product is determined, as usual, with the help of the formula

$$(\mathbf{u}, \mathbf{v}) = \int_D (u_1v_1 + u_2v_2 + u_3v_3)\,d\Omega.$$

In this space, we shall consider functions which are continuous and continuously differentiable in D, and satisfy one of the conditions (1.2), as well as the inequality $\int_D |A\mathbf{u}|^2 d\Omega < \infty$.

We form the scalar product

$$(A\mathbf{u}, \mathbf{u}) = \int_D \mathbf{u} A \mathbf{u} \, d\Omega. \tag{1.4}$$

According to Betti's third formula (4.26′), Ch. 2, Vol. 1 (considering that the displacements satisfy an inhomogeneous equation), we arrive at the equality

$$\int_D \mathbf{u} A \mathbf{u} \, d\Omega = 2 \int_D W(\mathbf{u}) \, d\Omega - \int_S \mathbf{u} T_\nu \mathbf{u} \, dS.$$

The last integral vanishes in view of the homogeneity of the boundary conditions (1.2). As a result, we arrive at the equality

$$\int_D \mathbf{u} A \mathbf{u} \, d\Omega = 2 \int_D W(\mathbf{u}) \, d\Omega, \tag{1.5}$$

from which it follows that the operator under consideration is symmetric.

Since $W(\mathbf{u})$ is a positive definite quadratic form, it follows from the equality $\int_D \mathbf{u} A \mathbf{u} \, d\Omega = 0$ that all deformations in the body vanish. Hence, the displacements are determined to within the rigid displacement. However, it follows from the condition at the surface S_1 (if it does not degenerate) that the displacements become identically equal to zero. Thus, for the case of the first and mixed problems, it is shown that the operator A is positive. Consequently, the operator of the theory of elasticity has an equivalent variational interpretation for these problems: the function which is the solution of this operator minimizes the functional

$$F(\mathbf{u}) = (A\mathbf{u}, \mathbf{u}) - 2(\mathbf{P}, \mathbf{u}) = 2 \int_D [W(\mathbf{u}) - \mathbf{P} \cdot \mathbf{u}] \, d\Omega \tag{1.6}$$

in a set of functions satisfying the first of the conditions (1.2).

Complications arising in the case of the second problem are identical to the Neumann problem for the Laplace equation. In order to obtain a positive operator in this case as well, the following restriction is imposed on the space of allowed functions:

$$\int_D \mathbf{u} \, d\Omega = 0, \qquad \int_D (\mathbf{r} \times \mathbf{u}) \, d\Omega = 0. \tag{1.7}$$

As a matter of fact, it follows from the condition $(\mathbf{u}, A\mathbf{u}) = 0$ that the displacements are determined to within a rigid displacement. However, if they are subjected to the restrictions (1.7) as well, we find that they vanish.

On the other hand, analysis of the second interior problem is possible only when the conditions of equilibrium of the body as a whole are satisfied. In this case, in view of the homogeneity of the boundary conditions and inhomogeneity of the equation itself, these equations assume the form [1]

[1] Conversely, for a homogeneous equation and inhomogeneous boundary conditions, the restrictions assume the form (1.7), Ch. 3, Vol. 1.

$$\int_D \mathbf{P}\, d\Omega = 0, \quad \int_D (\mathbf{r} \times \mathbf{P})\, d\Omega = 0. \tag{1.8}$$

Hence, all constructions will be carried out in a narrower Hilbert space, and we shall require that each of its elements should satisfy the second condition in (1.2) ($S_2 = S$), as well as the conditions (1.7). It will now follow from the condition $(\mathbf{u}, A\mathbf{u}) = 0$ that $\mathbf{u} \equiv 0$. Thus, the operator of the theory of elasticity is found to be positive.

In the general case of the mixed problem (for inhomogeneous boundary conditions), we arrive at the functional

$$F(\mathbf{u}) = 2\left\{\int_D [W(\mathbf{u}) - \mathbf{u}\cdot\mathbf{P}]\, d\Omega - \int_{S_2} \mathbf{u}\cdot\mathbf{F}_2\, dS\right\}, \tag{1.9}$$

where $\mathbf{F}_2$ is the stress vector which is specified on the surface S_2. In particular cases, we have

$$F(\mathbf{u}) = 2\int_D [W(\mathbf{u}) - \mathbf{u}\cdot\mathbf{P}\, d\Omega \quad \text{for} \quad S = S_1, \tag{1.9'}$$

$$F(\mathbf{u}) = 2\left\{\int_D [W(\mathbf{u}) - \mathbf{u}\cdot\mathbf{P}]\, d\Omega - \int_S \mathbf{u}\cdot\mathbf{F}_2\, dS\right\} \quad \text{for} \quad S = S_2. \tag{1.9''}$$

The boundary conditions can be satisfied only on the part of the surface where the displacements are given, since the boundary conditions in stresses are natural.

It should be noted that the result established above about the minimum of a functional is the well-known principle of minimum potential energy in the theory of elasticity. According to this principle, of all the displacements which satisfy the boundary conditions for them, only such displacements are actually realized for which the potential energy is minimum.

In accordance with the general procedure of investigation of variational formulation of problems (Sec. 12, Ch. 1, Vol. 1), we require a more rigorous result than the positive nature of the corresponding operator if we have to prove the solvability of these problems. In other words, we must prove the positive definiteness of this operator.

Let us consider the first basic problem [134]. In order to simplify the notation, we introduce Greek letters for the case of summation over repeated indices, and the summation will take place even when there is no repeated index, although in this case the corresponding term is squared.

We introduce the notation

$$s_{km} = \frac{1}{2}(u_{k,m} + u_{m,k}), \quad r_{km} = \frac{1}{2}(u_{k,m} - u_{m,k}),$$

$$H(\mathbf{u}) = \int_D u_\alpha^2\, d\Omega, \quad D(\mathbf{u}) = \int_D u_{\alpha,\beta}^2\, d\Omega, \tag{1.10}$$

$$S(\mathbf{u}) = \int_D s_{\alpha\beta}^2\, d\Omega, \quad R(\mathbf{u}) = \int_D r_{\alpha\beta}^2\, d\Omega.$$

The following equality can be proved without any difficulty:

$$D(\mathbf{u}) = R(\mathbf{u}) + S(\mathbf{u}). \tag{1.11}$$

In order to prove the positive definiteness of the operator of the theory of elasticity, we must derive the equality

$$R(\mathbf{u}) \leqslant cS(\mathbf{u}) \quad (c = \text{const}), \tag{1.12}$$

called Korn's inequality. It should be recalled that since we are considering the first problem now, the inequality will be proved under the condition $\mathbf{u}|_S = 0$. In this case,

$$D(\mathbf{u}) \leqslant (1 + c)S(\mathbf{u}). \tag{1.13}$$

We shall assume that the displacement $\mathbf{u}$ and its first and second derivatives are continuous right up to the boundary surface, which may even be piecewise smooth. The following equalities are obviously valid:

$$u_{k,m} = s_{km} + r_{km}, \quad u_{m,k} = s_{km} - r_{km}.$$

Multiplying these equalities and summing over both the indices, we get

$$u_{\alpha,\beta}u_{\beta,\alpha} = s_{\alpha\beta}^2 - r_{\alpha\beta}^2.$$

Since $u_{\alpha,\alpha}u_{\beta,\beta} = s_{\alpha\alpha}^2$, we get

$$s_{\alpha\beta}^2 - r_{\alpha\beta}^2 - s_{\alpha\alpha}^2 = u_{\alpha,\beta}u_{\beta,\alpha} - u_{\alpha,\alpha}u_{\beta,\beta} = (u_\alpha u_{\beta,\alpha})_{,\beta} - (u_\alpha u_{\beta,\beta})_{,\alpha}. \tag{1.14}$$

Let us now take the volume integral of the left- and right-hand sides of Eq. (1.14). We transform the integral of the right-hand side into a surface integral. In view of the homogeneity of the boundary conditions, we find that this integral is equal to zero. Hence the integral of the left-hand side is also equal to zero:

$$\int_D (s_{\alpha\beta}^2 - r_{\alpha\beta}^2 - s_{\alpha\alpha}^2)\,d\Omega = 0: \tag{1.15}$$

We then get

$$\int_D r_{\alpha\beta}^2 d\Omega = \int_D (s_{\alpha\beta}^2 - s_{\alpha\alpha}^2)\,d\Omega \leqslant \int_D s_{\alpha\beta}^2\,d\Omega. \tag{1.16}$$

Using the notation introduced in Eq. (1.16), we find that this inequality is Korn's inequality, and $c = 1$.

Since $W(\mathbf{u})$ is a positive definite quadratic form of strains, it will also be a positive definite quadratic form of the quantity s_{km}. Denoting by m_0 the lowest value of this quadratic form, we find that $W(\mathbf{u}) \geqslant m_0 s_{\alpha\beta}^2$.

Consequently,

$$\int_D W(\mathbf{u})\,d\Omega \geqslant m_0 \int_D s_{\alpha\beta}^2\,d\Omega = m_0 S(\mathbf{u}). \tag{1.17}$$

With the help of the inequality (1.13), we get

$$\int_D W(\mathbf{u})\,d\Omega \geqslant \frac{m_0}{2} D(\mathbf{u}). \tag{1.18}$$

Next, we use Friedrich's inequality (11.43), Ch. 1, Vol. 1, and apply it to each component of the displacement vector (this can be done, since the displacements on the surface vanish):

$$\int_D u_k^2 d\Omega \leqslant \frac{1}{\varkappa} \int_D u_{k,\beta}^2 d\Omega. \tag{1.19}$$

Adding these inequalities for each component, we get

$$H(\mathbf{u}) = \int_D u_\alpha^2 d\Omega \leqslant \frac{1}{\varkappa} D(\mathbf{u}). \tag{1.20}$$

Summing all the inequalities obtained above, we arrive at the required statement

$$(A\mathbf{u}, \mathbf{u}) \geqslant \gamma^2 H(\mathbf{u}) = \gamma^2 \|\mathbf{u}\|^2 \quad \left(\gamma^2 = \frac{m_0 \varkappa}{2}\right). \tag{1.21}$$

Let us now consider the second basic problem. Since the proof is cumbersome, we shall not derive Korn's inequality required in this case [135, 136] under the conditions

$$\int_D r_{km} d\Omega = 0. \tag{1.22}$$

We apply Poincaré's inequality (11.45), Ch. 1, Vol. 1, to each component of displacements:

$$\int_D u_k^2 d\Omega \leqslant A \int_D u_{k,\alpha}^2 d\Omega. \tag{1.23}$$

In view of the conditions (1.7), the second integral on the right-hand side vanishes. Evaluating the integral (1.23), we get the inequality

$$\|\mathbf{u}\|^2 \leqslant AD(\mathbf{u}). \tag{1.24}$$

We take the origin of coordinates at the centre of gravity of the volume D, and introduce the vectors $\mathbf{b}^{(km)}$ with components

$$b_k^{(km)} = x_m, \; b_m^{(km)} = -x_k, \; b_i^{(km)} = 0 \quad (i \neq k, i \neq m). \tag{1.25}$$

These vectors correspond to the rotation of the body as a rigid entity.

The inequality (1.24) is applicable to the displacement $\mathbf{u}^* = \mathbf{u} + c_{km}\mathbf{b}^{(km)}$, where c_{km} are arbitrary constants. In addition, we require that the displacement $\mathbf{u}^*$ (through a suitable choice of the constants c_{km}) should satisfy the equality (1.22). It should be noted that the conditions (1.7) are not violated as we go over to $\mathbf{u}^*$.

From Korn's inequality, we now get

$$R(\mathbf{u}^*) \leqslant CS(\mathbf{u}^*)$$

and hence

$$\frac{\|\mathbf{u}^*\|^2}{A} \leqslant D(\mathbf{u}^*) \leqslant \frac{1+C}{m_0} \int_D W(\mathbf{u}^*) \, d\Omega$$

$$= \frac{1+C}{m_0} \int_D W(\mathbf{u}) \, d\Omega = (A\mathbf{u}, \mathbf{u}^*). \tag{1.26}$$

It remains to be shown that $\|\mathbf{u}^*\|^2 \geqslant \|\mathbf{u}\|^2$. This inequality holds in view of the second equation in (1.7).

We thus arrive at the required result:

$$(A\mathbf{u}, \mathbf{u}) \geqslant \gamma^2\|\mathbf{u}\|^2 \quad \left(\gamma = \sqrt{\frac{m_0}{A(1 + C)}}\right). \tag{1.27}$$

It has been shown in [137] that the operator of the theory of elasticity is positive definite in the case of mixed problem as well.

In accordance with the general theory, we arrive at the following statement: in the energy space, there always exists a solution (normally, a generalized solution) of variational problems corresponding to the basic and mixed problems of the theory of elasticity. This solution may be obtained by the Ritz method.

We shall now touch upon the question as to what are the conditions under which the generalized solution is a solution in the classical sense (i.e. has the required number of derivatives, satisfies the differential equation, as well as the boundary conditions). It is found [138, 139] that if the right-hand side is a piecewise continuous, Lamé's equations are satisfied. The boundary conditions are satisfied if the boundary surface is sufficiently smooth, and the right-hand sides of the equilibrium equations are continuously differentiable a sufficient number of times. In the general case, the homogeneous boundary conditions are satisfied in the following sense. There exists a sequence of functions $\mathbf{u}_n$ (appearing in the energy space), such that the following equality is satisfied:

$$\lim_{n \to \infty} \int_D \sum_{i,k=1}^{3} \left(\frac{\partial u_{ni}}{\partial x_k} - \frac{\partial u_{0i}}{\partial x_k}\right) d\Omega = 0, \tag{1.28}$$

where $\mathbf{u}_0$ is a function minimizing the functional (i.e. the generalized solution).

Let us now consider the method of orthogonal projections (see Sec. 12, Ch. 1, Vol. 1). We shall start from the Hilbert space K of tensors $\mathbf{T}$ having determined the scalar product in the form of the integral

$$(\mathbf{T}', \mathbf{T}'') = \int_D (\sigma_x'\varepsilon_x'' + \sigma_y'\varepsilon_y'' + \sigma_z'\varepsilon_z'' + \tau_{xy}'\gamma_{xy}'' + \tau_{xz}'\gamma_{xz}'' + \tau_{yz}'\gamma_{yz}'')\,d\Omega. \tag{1.29}$$

The square of the norm is then equal to the integral

$$\|\mathbf{T}\|^2 = \int_D (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\,d\Omega \tag{1.30}$$

and is equal to twice the potential energy.

Let us directly go over to the mixed problem by way of an example (i.e. we proceed from the conditions (1.2)). We consider a set of tensors satisfying the homogeneous equilibrium equations, as well as the first of the conditions (1.2). We denote this set by K_1.[2] Next, we form a set K_2 of tensors satisfying the strain com-

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[2] In the case of the second basic problem ($S = S_2$), no boundary conditions are imposed on the set K_1.

patibility relations in stresses (Beltrami-Michell equations, see Sec. 4, Ch. 2, Vol. 1), where the corresponding displacements must satisfy first of the conditions (1.2).

Thus, two subspaces have been constructed in the space K. We shall now show that the entire space K is a direct sum of the spaces K_1 and K_2, i.e.

$$K = K_1 \oplus K_2. \tag{1.31}$$

We are now required to show that any two tensors $\mathbf{T}_1 \in K_1$ and $\mathbf{T}_2 \in K_2$ are orthogonal, and that any tensor $\mathbf{T}$ may be represented in the form $\mathbf{T}_1 + \mathbf{T}_2$ (where $\mathbf{T}_1 \in K_1$ and $\mathbf{T}_2 \in K_2$).

In the expression (1.29) (into which the tensors $\mathbf{T}_1$ and $\mathbf{T}_2$ have been substituted), we replace the strains ε_{ij}^1 by displacements and integrate the equation by parts. This gives

$$(\mathbf{T}_1, \mathbf{T}_2) = \int\limits_D \left[u_x^1 \left(\frac{\partial \sigma_x^2}{\partial x} + \frac{\partial \tau_{xy}^2}{\partial y} + \frac{\partial \tau_{xz}^2}{\partial z} \right) + u_y^1 \left(\frac{\partial \tau_{xy}^2}{\partial x} + \frac{\partial \sigma_y^2}{\partial y} + \frac{\partial \tau_{yz}^2}{\partial z} \right) \right.$$

$$\left. + u_z^1 \left(\frac{\partial \tau_{xz}^2}{\partial x} + \frac{\partial \tau_{yz}^2}{\partial y} + \frac{\partial \sigma_z^2}{\partial z} \right) \right] + \int\limits_S \mathbf{u}^1 \mathbf{T}_\nu^2 \mathbf{u} DS. \tag{1.32}$$

Since the tensor $\mathbf{T}_2$ satisfies homogeneous equilibrium conditions,the volume integral in (1.32) vanishes. The surface integral also vanishes in view of the boundary conditions. Thus, the orthogonality of the subspaces K_1 and K_2 is proved.

We shall now show that any tensor $\mathbf{T}$ may be represented in the form of the sum $\mathbf{T}_1 + \mathbf{T}_2$ (it is assumed that the components of the tensor $\mathbf{T}$ are continuous and continuously differentiable functions in D). We denote by $\mathbf{T}_1$ the stress tensor satisfying the homogeneous conditions (1.2), as well as the equilibrium equations. Then, $\mathbf{T}_2 = \mathbf{T} - \mathbf{T}_1$.

Suppose that $\mathbf{T}_0$ is a tensor which is a solution of the problem of the theory of elasticity. We denote by $\tilde{\mathbf{T}}$ the tensor satisfying the equilibrium equations, as well as the second of the conditions (1.2). Obviously the tensor $\mathbf{T}_0$ is the projection of the tensor $\tilde{\mathbf{T}}$ on the subspace K_1. Since $(\tilde{\mathbf{T}} - \mathbf{T}_0) \in K_2$, we get $(\mathbf{T}_0, \tilde{\mathbf{T}} - \mathbf{T}_0) = 0$. In this case,

$$\|\tilde{\mathbf{T}}\|^2 = \|\mathbf{T}_0\|^2 + \|\tilde{\mathbf{T}} - \mathbf{T}_0\|^2 \geqslant \|\mathbf{T}_0\|^2. \tag{1.33}$$

The inequality (1.33) may be interpreted in the following way (Castigliano's principle): of all the stress tensors satisfying the equilibrium equations as well as the boundary condition in stresses, it is the real stresses that impart the minimum potential energy to the body.

Let us now go over to a consideration of problems in which restrictions have been imposed on the basis of variational inequalities. As mentioned in Sec. 1, Ch. 3, Vol. 1, this kind of problems arise when a rigid body comes in contact with elastic bodies, or when elastic bodies come in contact with one another. It should be observed that practically we are speaking of a rigid surface (since the length of the rigid body itself is immaterial). We assume that there are no shearing stresses on the

surface of the elastic body. The equation for the rigid surface is given by

$$\psi(q) = 0. \tag{1.34}$$

This equation permits us to express in a convenient form the condition of non-penetration of the elastic body through the surface. Consequently, after linearization of this condition, we arrive at the inequality (1.14), Ch. 3, Vol. 1:

$$\psi(q) + \operatorname{grad} \psi(q) \cdot \mathbf{u}(q) \geqslant 0.$$

We shall assume that outside the contact surfaces the normal stresses vanish, and that they are not positive inside the contact surface. Then, repeating the arguments used while considering the analogous problem for a harmonic function, we arrive at the variational inequality [140]

$$\int_D \Delta^* \mathbf{u} \Delta^* (\mathbf{v} - \mathbf{u})\, d\Omega \geqslant \int_D \mathbf{P} \cdot (\mathbf{v} - \mathbf{u})\, d\Omega, \tag{1.35}$$

which is valid for any displacement $\mathbf{v}$ satisfying the condition

$$\psi(q) + \operatorname{grad} \psi(q) \cdot \mathbf{v}(q) \geqslant 0 \qquad (q \in S).$$

The function $\mathbf{u}$ satisfying the inequality (1.35) is also the solution of the above-mentioned problem with constraints.

Using Korn's inequality (1.12), we can show that the problem with constraints (under specified conditions) has been reduced to the problem of minimization of the functional (1.3) under the condition imposed above on the function $\mathbf{u}$.

In order to solve this problem, all the methods described in Sec. 12, Ch. 1, Vol. 1 are applicable. In particular, if we introduce the Lagrangian functional (Arrow-Gurvits algorithm) [141]

$$\mathscr{L}(\mathbf{u}, p) = F(\mathbf{u}) - \int_{S_1} [\psi(q) + \operatorname{grad} \psi(q) \cdot \mathbf{u}(q)] p(q)\, dS_q, \tag{1.36}$$

the basic stage of the algorithm is realized by the following recurrent process [3]:

$$p^{k+1} = P_k \{p^k - \rho_k[\psi(q) + \operatorname{grad} \psi(q) \cdot \mathbf{u}^k(q)]\}, \tag{1.37}$$

where the operator P_k acts as follows:

$$P_k f = \begin{cases} 0, & f > 0, \\ f, & f \leqslant 0. \end{cases} \tag{1.38}$$

Let us now consider the actual realization of the Ritz method in the problems of the theory of elasticity. It should be noted that the construction of coordinate functions is a fairly complicated problem. In some cases (sphere, parallelepiped, etc.) this problem is solved comparatively easily [142].

We shall mention another fairly general approach on the basis of the theory of R-functions [143].

As mentioned in Sec. 12, Ch. 1, Vol. 1, the solution of boundary value problems by the Ritz method may lead to an unstable algorithm. We shall illustrate this state-

[3] It can be shown that $p(q)$ is proportional to the normal pressure distributed over the contact surface.

ment by taking as an example the problem of bending of a plate in the form of a circular sector under mixed boundary conditions [144].

Let a and b be the outer and the inner radius respectively, $2\varphi_0$ be the angle of the sector. The problem consists in determining the displacement $w(x, y)$ satisfying Eq. (4.25), Ch. 3, Vol. 1, under the boundary conditions

$$w(x, y)\Big|_{L_1} = 0, \qquad \frac{\partial w(x, y)}{\partial n}\Big|_{L_1} = 0$$

on a part L_1 of the contour, and

$$M_n\Big|_{L_2} = 0, \qquad \left[Q_n + \frac{\partial M_{n\tau}}{\partial s}\right]\Big|_{L_2} = 0$$

on the remaining part L_2.

It is not difficult to show (in accordance with the general scheme) that in the variational formulation (for q = const), the problem can be reduced to finding the minimum of the functional

$$\iint\Bigg[\left(\frac{\partial^2\tilde{w}}{\partial x^2}\right)^2 + \left(\frac{\partial^2\tilde{w}}{\partial y^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2\tilde{w}}{\partial x\,\partial y}\right)^2$$

$$+ 2\nu\left(\frac{\partial^2\tilde{w}}{\partial x\,\partial y}\right)^2 + 2\nu\frac{\partial^2\tilde{w}}{\partial x^2}\frac{\partial^2\tilde{w}}{\partial y^2} - \frac{2}{a^2}\tilde{w}\Bigg]\,dx\,dy, \qquad (1.39)$$

where $\tilde{w} = Dw/qa^2$, and D is the rigidity of the plate.

The solution will be constructed in polar coordinates r, $\varphi (b < r < a,\ |\varphi| \leqslant$ $\leqslant (1/2)\,\varphi_0)$ by going over to the dimensionless variables

$$\rho = \frac{r-b}{a}\ \left(0 \leqslant \rho \leqslant \rho_0 = \frac{a-b}{a}\right), \quad \tilde{\varphi} = \frac{\varphi - (1/2)\,\varphi_0}{\varphi_0}\ \ (|\tilde{\varphi}| \leqslant 1/2). \quad (1.40)$$

We shall assume that the plate is rigidly fastened $\left(\tilde{w} = \dfrac{\partial\tilde{w}}{\partial n} = 0\right)$ on the internal arc $(\rho = 0)$, while it is freely supported on the remaining regions of the contour. Since the fastening conditions are the main conditions, the coordinate functions must satisfy them. We shall proceed from two systems of coordinate functions. The functions of the first system will be chosen in the form of the products,

$$w_{ij} = \rho^{i+2}\tilde{\varphi}^{2j} \quad (i, j = 0, 1, 2, \dots). \qquad (1.41)$$

The functions of the second system are obtained from the first one in the following manner: we orthogonalize the functions ρ^i on the segment $(0, \rho_0)$, and the functions $\tilde{\varphi}^{2j}$ on the segment $(-1/2, 1/2)$. We denote the functions thus obtained through $\bar{\rho}_i$ and $\bar{\varphi}_j$. The product of these functions constitutes the second system of coordinate functions

$$\tilde{w}_{ij} = \bar{\rho}_i\bar{\varphi}_j \quad (i, j = 0, 1, 2, \dots). \qquad (1.42)$$

It can be shown that the functions of the first coordinate system are not strongly

minimal in the corresponding energy space, while the functions of the second system are strongly minimal.

Calculations were carried out for parameters $\rho_0 = 1.82$ and $\varphi_0 = 48°$. In all, 12 coordinate functions ($i = 0, 1, 2, 3$ and $j = 0, 1, 2$) were used. The solution of the system of equations was carried out nearly precisely, although some error was introduced while constructing the matrix itself. We denote by δ_a and δ_w the relative error introduced in the Ritz coefficients and in the displacement w, starting from the first coordinate system. The bars indicate that the same quantities were taken but they are now applied to the second system, i.e. $\bar{\delta}_a$ and $\bar{\delta}_w$.

We shall now give the values of these quantities, reducing them to the error $\delta_0 = 10^{-7}$ (artificially given) for the coefficients of the Ritz matrix (see Table 19).

TABLE 19

i, j	0, 0	1, 0	0, 1	1, 1	0, 2
δ_a/δ_0	875	3930	54 400	12 680	13 000
$\bar{\delta}_a/\delta_0$	0.195	0.692	0.550	0.314	2.82

i, j	2, 1	1, 2	2, 2	3, 0	3, 2
δ_a/δ_0	34 700	16 900	20 200	23 600	22 900
$\bar{\delta}_a/\delta_0$	0.67	0.47	0.595	39.7	0.957

Thus, in the case of non-minimal systems, the errors in the coefficients (and hence in the solutions) are extremely large. It should be noted that on the outer arc, the ratio δ_w/δ_0 varies between 36 and 82. At the same time, the ratio $\bar{\delta}_w/\delta_0$ varies between 0.1 and 0.23.

Section 2
Construction of Minimizing Sequences. The Method of Finite Elements

As before, let an isotropic elastic body occupy a domain D bounded by the surface $S = S_1 \cup S_2$, but in this case, the zero vector displacement $\mathbf{F}_1 = 0$ is given on S_1, and the stress vector $\mathbf{F}_2$ is given on S_2. The vector of body forces $\mathbf{P}$ acts in the domain D. It is required to determine the state of stress and strain in the domain D.

Let us recall the expression for the elastic potential (3.23), Ch. 2, Vol. 1:

$$W = \frac{1}{2}(\varepsilon_x\sigma_x + \varepsilon_y\sigma_y + \varepsilon_z\sigma_z + \gamma_{xz}\tau_{xz} + \gamma_{yz}\tau_{yz} + \gamma_{xy}\tau_{xy}) = \frac{1}{2}\varepsilon^{\mathrm{T}}\sigma, \quad (2.1)$$

where the superscript T denotes transposition.

In the variational approach (as has been shown in Sec. 1), our problem is reduced to the minimization of the quadratic functional

$$F(\mathbf{u}) = (A\mathbf{u}, \mathbf{u}) - 2(\mathbf{P}, \mathbf{u}) - 2\int_{S_2} \mathbf{u} \cdot \mathbf{F}_2 dS$$

$$= \int_D \mathbf{u}^{\mathrm{T}} A \mathbf{u}\, d\Omega - 2 \int_D \mathbf{P}^{\mathrm{T}} \cdot \mathbf{u}\, d\Omega - 2 \int_{S_2} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{F}_2\, dS. \tag{2.2}$$

It has been shown earlier (see Sec. 1) that this functional coincides with the expression for the total potential energy

$$U = \int_D W\, d\Omega - \int_D \mathbf{P}^{\mathrm{T}} \cdot \mathbf{u}\, d\Omega - \int_{S_2} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{F}_2 dS.$$

In order to minimize the functional U, we apply the apparatus of the method of finite elements, described in Sec. 13, Ch. 1, Vol. 1. This functional can be rewritten in the following form:

$$U = \frac{1}{2} \int_D \varepsilon^{\mathrm{T}} \sigma\, d\Omega - \int_D \mathbf{u}^{\mathrm{T}} \cdot \mathbf{P}\, d\Omega - \int_{S_1} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{F}_2 dS. \tag{2.3}$$

Here, the first integral is a quadratic form with respect to the strain components ε, since in accordance with Hooke's law (3.30), Ch. 2, Vol. 1, $\sigma = C\varepsilon$, where the matrix C is equal to

$$C = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{2.4}$$

As in the one-dimensional case (see Sec. 13, Ch. 1, Vol. 1), the vector equation is of the second order once again, and the first integral in the expression for U is connected with the derivatives of the displacement vector. The last two integrals on the right-hand side of (2.3) are connected with the loading of the body by mass and surface forces. In the one-dimensional case, the last integral was missing on account of zero boundary conditions. Here, it appears as a result of the application of the Gauss-Ostrogradskii formula to the volume integrals.

Obviously, by dividing a body into finite elements of appropriate dimensions and specifying piecewise polynomial vector functions on the finite elements for approximating the vector $\mathbf{u}$, we can carry out the same calculations as in the case of differential equation (13.1), Ch. 1, Vol. 1, and obtain the following approximation for U:

$$U(\mathbf{u}^h) = \frac{1}{2} \mathbf{q}^{\mathrm{T}} K \mathbf{q} - \mathbf{F}^{\mathrm{T}} \cdot \mathbf{q}. \tag{2.5}$$

Here, K is the rigidity matrix of the system, $\mathbf{q}$ is the vector for nodal unknowns (displacements), and the vector $\mathbf{F}$ is the load from body and surface forces, reduced to the nodes. In order to construct the global matrix and the global vectors, it is sufficient to calculate the corresponding objects of one finite element and to carry out the summation after placing these objects at the appropriate places in the global massive. This summation is accomplished by formal calculations (similar methods are used, for example, for constructing the equilibrium equations of a system of bars in structural mechanics [145]).

It is now necessary to construct matrices and vectors pertaining to one element.

Let a polynomial approximation be given on the element in the following form:

$$\mathbf{u}^e = \mathbf{V}^e \times \mathbf{q}^e. \tag{2.6}$$

Here, the symbol e indicates that the function or the vector belongs to some element, and $\mathbf{q}$ is the vector of unknowns at the nodal points including the displacements at these points and, if necessary, their derivatives. The strain vector is given by

$$\varepsilon^e = A\mathbf{u}^e = A\mathbf{V}^e \times \mathbf{q}^e = B^e\mathbf{q}^e \qquad (B^e = A\mathbf{V}^e), \tag{2.7}$$

where A denotes the operator

$$A = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{pmatrix}. \tag{2.8}$$

We can now represent the stress vector in the form

$$\sigma^e = C\varepsilon^e = CB^e \cdot \mathbf{q}^e.$$

The integral $\int\limits_{D^e} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{P}\, d\Omega$ can be written in the following form upon approximation of the functional U on the element:

$$\int\limits_{D^e} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{P}\, d\Omega = \mathbf{q}^{e^{\mathrm{T}}} \int\limits_{D^e} \mathbf{V}^{e^{\mathrm{T}}} \cdot \mathbf{P}\, d\Omega = \mathbf{q}^{e^{\mathrm{T}}} \cdot \mathbf{p}^e. \tag{2.9}$$

Here, $\mathbf{p}^e$ denotes the vector of nodal forces equivalent to the body forces (an exactly similar approach was adopted in the one-dimensional problem). If some element has a common face with the boundary S_2 on which the vector $\mathbf{F}_2$ of the surface forces is given, it is necessary to calculate the following integral for this element:

$$\int\limits_{S_2^e} \mathbf{V}^{e^{\mathrm{T}}} \cdot \mathbf{F}_2\, dS = \mathbf{q}^{e^{\mathrm{T}}} \int\limits_{S_2^e} \mathbf{V}^{e^{\mathrm{T}}} \cdot \mathbf{F}_2\, dS = \mathbf{q}^{e^{\mathrm{T}}} \cdot \mathbf{r}^e, \tag{2.10}$$

where $\mathbf{r}^e$ denotes the vector of nodal forces equivalent to the surface forces. The rigidity matrix K^e of the element appears as a result of the approximation of the integral $\frac{1}{2} \int\limits_{D} \varepsilon^{\mathrm{T}}\sigma\, d\Omega$ on the element:

$$\frac{1}{2}\int_{D^e}\varepsilon^{\mathrm{T}}\sigma\,d\Omega = \frac{1}{2}\int_{D^e}(B^e\mathbf{q}^e)^{\mathrm{T}}\,CB^e\,\mathbf{q}^e\,d\Omega$$

$$= \frac{1}{2}\,\mathbf{q}^{e^{\mathrm{T}}}\int_{D^e}B^{e^{\mathrm{T}}}CB^e\,\mathbf{q}^e\,d\Omega = \frac{1}{2}\,\mathbf{q}^{e^{\mathrm{T}}}K^e\mathbf{q}^e. \quad (2.11)$$

Thus, the approximation of the quadratic functional on a finite element is given by

$$U^e = \frac{1}{2}\,\mathbf{q}^{e^{\mathrm{T}}}K^e\mathbf{q}^e - \mathbf{q}^{e^{\mathrm{T}}}\cdot\mathbf{p}^e - \mathbf{q}^{e^{\mathrm{T}}}\cdot\mathbf{r}^e. \quad (2.12)$$

Carrying out summation over all the elements, we get

$$U^h = \frac{1}{2}\,\mathbf{q}^{\mathrm{T}}K\mathbf{q} - \mathbf{q}^{\mathrm{T}}\cdot p - \mathbf{q}^{\mathrm{T}}\cdot\mathbf{r} = \frac{1}{2}\,\mathbf{q}^{\mathrm{T}}K\mathbf{q} - \mathbf{q}^{\mathrm{T}}\cdot\mathbf{F} \qquad (\mathbf{F} = \mathbf{p} + \mathbf{r}). \quad (2.13)$$

Minimization of U^h leads to the system of equations for the method of finite elements:

$$K\mathbf{q} = \mathbf{F}. \quad (2.14)$$

The boundary conditions on a part of the surface S_1 may be considered by neglecting the given components of the vector $\mathbf{q}$, in the corresponding rows and columns in the matrix K, as well as the appropriate components of the vector $\mathbf{F}$.

Let us find out the mechanical meaning of the coefficients of the rigidity matrix. For the sake of convenience, we shall consider a system having one degree of freedom at each node (to each component of the vector $\mathbf{q}$, there corresponds one degree of freedom). We assume that a set of nodal loads has been applied to the system in such a way that it causes a displacement of the jth node by unity, while the rest of the nodes remain stationary, i.e.

$$q_j = 1, \qquad q_k = 0 \qquad (k \neq j).$$

The non-zero component of the vector $\mathbf{q}$ is multiplied only by the coefficients K_{ij} ($i = 1, 2, \ldots, N$), and so the system (2.14) assumes the form

$$K_{ij} = F_i. \quad (2.15)$$

Thus, the coefficients K_{ij} have the following meaning: they represent the reaction at the ith node upon a unit displacement of the jth node, when all the remaining nodes remain stationary. The matrix K is a banded matrix, and if the element $K_{ij} \neq 0$, it means that the nodes i and j are connected through some element. Hence, it may be stated that the displacements of the node i cause reactions at the node j only if they belong to the same element. A similar treatment is naturally also possible when the nodes have several degrees of freedom.

We shall write down the individual stages of calculation of the rigidity matrix of an element. First of all, we introduce an approximation for the displacement vector in the element in terms of the displacements (and, possibly, through their derivatives) at the nodes:

$$\mathbf{u}^e = \mathbf{V}^e \times \mathbf{q}^e.$$

Next, we find the matrix

$$B^e = A\mathbf{V}^e$$

and determine the rigidity matrix of the element

$$K^e = \int_{D^e} B^{e^{\mathrm{T}}} C B^e \, d\Omega.$$

The equivalent nodal load to the mass forces is equal to

$$\mathbf{p}^e = \int_{D^e} \mathbf{V}^{e^{\mathrm{T}}} \times \mathbf{P} \, d\Omega,$$

while the equivalent nodal load to the surface forces is

$$\mathbf{r}^e = \int_{S_2^e} \mathbf{V}^{e^{\mathrm{T}}} \times \mathbf{F}_2 dS.$$

We shall demonstrate the computation of the rigidity matrix by taking the example of a plane triangular element under linear interpolation. We consider a triangular element with nodes i, j, m, designated in the anticlockwise direction (Fig. 82). For the displacements, we get

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y, \quad v = \alpha_4 + \alpha_5 x + \alpha_6 y. \tag{2.16}$$

We shall now express the indeterminate coefficients $\alpha_1, \alpha_2, \ldots, \alpha_6$ in terms of the displacements at the nodes (here, $\mathbf{q}^e = \{u_i, v_i, u_j, v_j, u_m, v_m\}$). For this purpose, we solve the system of equations

$$\begin{aligned}
u_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i, \\
u_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j, \\
u_m &= \alpha_1 + \alpha_2 x_m + \alpha_3 y_m, \\
v_i &= \qquad\qquad\qquad\qquad \alpha_4 + \alpha_5 x_i + \alpha_6 y_i, \\
v_j &= \qquad\qquad\qquad\qquad \alpha_4 + \alpha_5 x_j + \alpha_6 y_j, \\
v_m &= \qquad\qquad\qquad\qquad \alpha_4 + \alpha_5 x_m + \alpha_6 y_m.
\end{aligned} \tag{2.17}$$

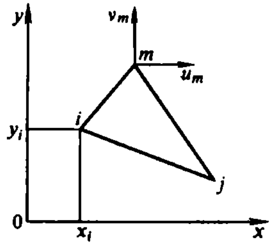

Fig. 82. A plane triangular element.

Then, for the element under consideration, we get the following expression for displacement u:

$$u = \frac{1}{2\Delta}[(a_i + b_i x + c_i y) u_i + (a_j + b_j x + c_j y) u_j + (a_m + b_m x + c_m y) u_m], \tag{2.18}$$

where

$$\begin{aligned} a_i &= x_j y_m - x_m y_j, \\ b_i &= y_j - y_m = y_{jm}, \\ c_i &= x_m - x_j = x_{mj}. \end{aligned} \tag{2.19}$$

The remaining coefficients are obtained by cyclic permutation of the indices, while

$$2\Delta = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}. \tag{2.20}$$

It should be noted that Δ is the area of the element.

Similarly, we can express v in terms of v_i, v_j, v_m.

Thus, the matrix $\mathbf{u}^e$ is found and, apparently, its dimensions are 2×6. The matrix operator A in this case is given by

$$A = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix}. \tag{2.21}$$

We then get

$$B^e = \frac{1}{2\Delta}\begin{pmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{pmatrix}. \tag{2.22}$$

In the case of the state of plane stress, the elasticity matrix C_1 assumes the form

$$C_1 = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}. \tag{2.4'}$$

Since the stresses and displacements do not change over the thickness (δ) of the plate, the rigidity matrix is calculated just by integration over the area of the element:

$$K = \delta \int_S B^{e^{\mathrm{T}}} C_1 B^e \, dx \, dy. \tag{2.23}$$

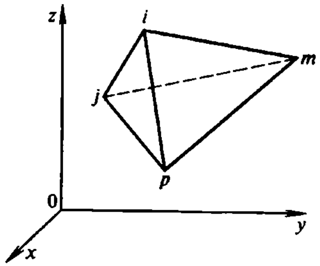

Fig. 83. A tetrahedral element.

The rigidity matrix may be written in the form

$$K = \begin{pmatrix} K_{ii} & K_{ij} & K_{im} \\ K_{ij} & K_{jj} & K_{jm} \\ K_{im} & K_{jm} & K_{mm} \end{pmatrix}, \tag{2.23'}$$

where each element K_{pq} is defined by a 2×2 block, and

$$K_{pq} = B_p^{e\mathrm{T}} C_1 B_q^e \Delta, \tag{2.24}$$

while $B_r^e (r = p, q)$ is a block of the matrix B^e:

$$B_r^e = \frac{1}{2\Delta} \begin{pmatrix} b_r & 0 \\ 0 & c_r \\ c_r & b_r \end{pmatrix}. \tag{2.25}$$

It is not difficult to obtain the values of the loads equivalent to the mass forces. For the force of gravity, in particular, we find that these are equally distributed between three nodes.

For a tetrahedral spatial element (Fig. 83), calculations are made in a similar manner. For the components u, v, w of the displacements, we get

$$u = \frac{1}{6\Delta_1} \sum_{q = i, j, m, p} (a_q + b_q x + c_q y + d_q z)\, u_q,$$

$$v = \frac{1}{6\Delta_1} \sum_{q = i, j, m, p} (a_q + b_q x + c_q y + d_q z)\, v_q, \tag{2.26}$$

$$w = \frac{1}{6\Delta_1} \sum_{q = i, j, m, p} (a_q + b_q x + c_q y + d_q z)\, w_q,$$

where

$$6\Delta_1 = \begin{vmatrix} 1 & x_i & y_i & z_i \\ 1 & x_j & y_j & z_j \\ 1 & x_m & y_m & z_m \\ 1 & x_p & y_p & z_p \end{vmatrix}, a_i = \begin{vmatrix} x_j & y_j & z_j \\ x_m & y_m & z_m \\ x_p & y_p & z_p \end{vmatrix}, \quad (2.27)$$

$$b_i = -\begin{vmatrix} 1 & y_j & z_j \\ 1 & y_m & z_m \\ 1 & y_p & z_p \end{vmatrix}, \quad c_i = -\begin{vmatrix} x_j & 1 & z_j \\ x_m & 1 & z_m \\ x_p & 1 & z_p \end{vmatrix}, \quad d_i = -\begin{vmatrix} x_j & y_j & 1 \\ x_m & y_m & 1 \\ x_p & y_p & 1 \end{vmatrix}.$$

Here, Δ_1 is the volume of the tetrahedron. The remaning coefficients are determined by a cyclic permutation of the indices. For a typical 3×3 block K_{pq} of the rigidity matrix, we have ($r = p, q$)

$$K_{pq} = B_p^{e\mathrm{T}} C B_q^e \Delta_1; \qquad B_r^e = \frac{1}{6\Delta_1}\begin{pmatrix} b_r & 0 & 0 \\ 0 & c_r & 0 \\ 0 & 0 & d_r \\ c_r & b_r & 0 \\ 0 & d_r & c_r \\ d_r & 0 & b_r \end{pmatrix} \quad (2.28)$$

In the two-dimensional case for a rectangle with four nodes, the approximation is constructed as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & x & y & xy & & & & \\ & & & & 1 & x & y & xy \end{pmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_8 \end{Bmatrix}. \quad (2.29)$$

For the three-dimensional case of a parallelepiped with eight nodes, the approximation is given by

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & x & y & z & xy & xz & yz & xyz & & & & & & & \\ & & & & & & & & 1 & x & y & \dots & & & \\ & & & & & & & & & & & & 1 & x & y & \dots \end{pmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_{24} \end{Bmatrix}.$$

(2.30)

We shall now consider an example of calculations by the method of finite elements. Figure 84 shows a division into elements in the problem of distribution of stresses around a circular hole under the conditions of homogeneous stress state away from the hole. The numerical solution obtained by the method of finite elements was compared with the analytical solution. The results of this comparison are shown in Fig. 85 (the solid lines show the exact solution, while the solution obtained by the method of finite elements is shown by points). As in the case of the one-dimensional problem, we can introduce an approximation with the help of Hermite polynomials, ensuring the continuity of the required number of derivatives in

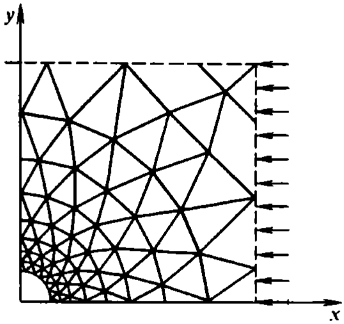

Fig. 84. Discretization for a domain with a circular cut.

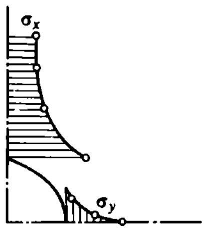

Fig. 85. A comparison between the exact solution and the solution obtained with the help of the finite elements method.

the solution. This permits a consideration of the problems for plates and shells[4]. There also exist methods which allow us to easily construct elements with interpolation by a polynomial of any degree (for this purpose, we use Lagrangian as well as certain other polynomials). The methods of approximation for elements with curvilinear boundaries are also known.

All the above description has so far been confined to static problems only. However, the method of finite elements allows a natural generalization to dynamic problems as well. We shall obtain an equation in this method for dynamic problems. If we use D'Alembert's principle, we must consider the equilibrium of a body under the action of all external forces, including the inertial forces. This means that the functional (2.2) will contain an additional term $\int_D \rho\, \mathbf{u}^T \ddot{\mathbf{u}}\, d\Omega$ (strictly

---

[4] As a matter of fact, the functionals corresponding to the problems for plates and shells contain derivatives of higher than the first order, and hence a linear interpolation leads to erroneous results.

speaking, the functional F is no longer the total potential energy, since it includes the kinetic energy). It is sufficient to calculate this additional integral just for one element. We have

$$\mathbf{u}^e(x, y, z, t) = \mathbf{V}^e(x, y, z) \times \mathbf{q}^e(t),$$
$$\int_{D^e} \rho \mathbf{u}^{\mathrm{T}} \ddot{\mathbf{u}}\, d\Omega = \mathbf{q}^{e^{\mathrm{T}}} \int_{D^e} \mathbf{V}^{e^{\mathrm{T}}} \mathbf{V}^e \ddot{\mathbf{q}}^e\, d\Omega = \mathbf{q}^{e^{\mathrm{T}}} M^e \ddot{\mathbf{q}}^e. \tag{2.31}$$

Here, $M^e = \int_{D^e} \rho \mathbf{V}^{e^{\mathrm{T}}} \mathbf{V}^e\, d\Omega$ is called the mass matrix of the element (similar to the mass matrix (13.28), Ch. 1, Vol. 1, in the one-dimensional case). The global matrix is obtained by adopting the same procedure as for the rigidity matrix. If we look for the stationary point of the functional F^h by considering the inertial forces, we now get a system of ordinary differential equations

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{F}. \tag{2.32}$$

To these, we must add the initial conditions

$$\mathbf{q}(0) = \mathbf{p}, \quad \dot{\mathbf{q}}(0) = \mathbf{r}. \tag{2.33}$$

The solution of the above system of equations is a non-trivial problem for the vector function $\mathbf{F}(t)$ of general type, while for higher-order systems, it is a more complicated problem. A common method of solving this problem is to construct a difference diagram in time. Dividing the interval of time under consideration into segments Δt and denoting the vector $\mathbf{q}$ at the ith node by $\mathbf{q}^i$, we can obtain an approximation for the second derivative. The simplest of these approximations is given by

$$\ddot{\mathbf{q}} = \frac{1}{\Delta t^2}(\mathbf{q}^{i+1} + \mathbf{q}^{i-1} - 2\mathbf{q}^i). \tag{2.34}$$

Substitution of this expression into the original system of differential equations leads to a system of algebraic equations for determining $\mathbf{q}^{i+1}$ from value of $\mathbf{q}^{i-1}$ and $\mathbf{q}^i$, which have already been calculated.

If the right-hand side is a function of the form $\mathbf{F}(t) = \mathbf{F}f(t)$, where $\mathbf{F}$ is a vector and $f(t)$ is smooth periodic function, it is better to construct the solution by expansion in eigenfunctions. Let us consider this case, which is frequently encountered, in greater details. For the sake of simplicity, we put $f(t) = \cos \lambda t$ and $\dot{\mathbf{q}}(0) = 0$. The general solution of the system

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{F}\cos \lambda t \tag{2.35}$$

is the sum of the particular solution of an inhomogeneous system and the general solution of a homogeneous system, which can be represented in the form

$$\mathbf{q}(t) = \sum_j A_j \mathbf{q}^{(j)} \cos \omega_j t. \tag{2.36}$$

Here, A_j are undetermined coefficients, and $\mathbf{q}^{(j)}$ and ω_j are the eigenvectors and eigenvalues of the equation

$$\boldsymbol{K}\mathbf{q} - \omega^2 \boldsymbol{M}\mathbf{q} = 0, \tag{2.37}$$

arranged in the increasing order of ω_j.

The eigenvectors have the property of biorthogonality [146]

$$q^{(i)\mathrm{T}} \boldsymbol{M}\mathbf{q}^{(j)} = 0 \qquad (i \neq j). \tag{2.38}$$

Besides, they can be normalized. This gives

$$\mathbf{q}^{(i)\mathrm{T}} \boldsymbol{M}\mathbf{q}^{(j)} = \delta_{ij}. \tag{2.39}$$

From this equation and (2.35), we can easily find that

$$\mathbf{q}^{(i)\mathrm{T}} \boldsymbol{K}\mathbf{q}^{(j)} = \omega_i^2 \delta_{ij}. \tag{2.40}$$

With the help of the relations (2.39) and (2.40), we can show that the function

$$\mathbf{q}^*(t) = \cos \lambda t \sum_j \mathbf{q}^{(j)} \frac{\mathbf{q}^{(j)\mathrm{T}} \cdot \mathbf{F}}{\omega_j^2 - \lambda^2} \tag{2.41}$$

is a particular solution of (2.35) for the case when $\lambda \neq \omega_j$ for all j. As a matter of fact, substituting $\mathbf{q}^*(t)$ into (2.35) and multiplying by $\mathbf{q}^{(i)\mathrm{T}}$, we get

$$-\mathbf{q}^{(i)\mathrm{T}} \lambda^2 \cos \lambda t \sum_j \boldsymbol{M}\mathbf{q}^{(j)} \frac{\mathbf{q}^{(j)\mathrm{T}} \cdot \mathbf{F}}{\omega_j^2 - \lambda^2}$$

$$+ \mathbf{q}^{(i)\mathrm{T}} \lambda^2 \cos \lambda t \sum_j \boldsymbol{K}\mathbf{q}^{(j)} \frac{q^{(j)\mathrm{T}} \cdot \mathbf{F}}{\omega_j^2 - \lambda^2} = \mathbf{q}^{(i)\mathrm{T}} \cdot \mathbf{F} \cos \lambda t. \tag{2.42}$$

Dividing both sides by $\cos \lambda t$ and using (2.39) and (2.40), we obtain the identity

$$-\lambda^2 \frac{\mathbf{q}^{(i)\mathrm{T}} \cdot \mathbf{F}}{\omega_i^2 - \lambda^2} + \omega_i^2 \frac{\mathbf{q}^{(i)\mathrm{T}} \cdot \mathbf{F}}{\omega_i^2 - \lambda^2} = \mathbf{q}^{(i)\mathrm{T}} \cdot \mathbf{F}. \tag{2.43}$$

Thus, the general solution of (2.35) is given by

$$\mathbf{q}(t) = \sum_j A_j \mathbf{q}^{(j)} \cos \omega_j t + \sum_j \mathbf{q}^{(j)} \frac{\mathbf{q}^{(j)\mathrm{T}} \cdot \mathbf{F}}{\omega_j^2 - \lambda^2} \cos \lambda t$$

$$= \sum_j \mathbf{q}^{(j)} \left(A_j \cos \omega_j t + \frac{\mathbf{q}^{(j)\mathrm{T}} \cdot \mathbf{F}}{\omega_j^2 - \lambda^2} \cos \lambda t \right). \tag{2.44}$$

Next, we expand the vector $\mathbf{p} = \mathbf{q}(0)$ in eigenvectors $\mathbf{q}^{(j)}$:

$$\mathbf{p} = \sum_j \mathbf{q}^{(j)} [\, \mathbf{q}^{(j)\mathrm{T}} \boldsymbol{M}\mathbf{p} \,]. \tag{2.45}$$

The validity of this equation can be easily verified by premultiplying it by $\mathbf{q}^{(j)\mathrm{T}}M$. Then, comparing $\mathbf{q}(0)$ from (2.44) and (2.45), we get

$$A_j = \mathbf{q}^{(j)\mathrm{T}} M\mathbf{p} - \frac{\mathbf{q}^{(j)\mathrm{T}} \cdot \mathbf{F}}{\omega_j^2 - \lambda^2}. \tag{2.46}$$

The number of terms in the sum (2.44) must be determined each time depending on the required accuracy and the properties of the problem under consideration (initial conditions, form of the right-hand side, etc.). In most cases, it is possible to manage with a small number of eigenvectors, and quite often, a single eigenvector is sufficient. Thus, for example, if $\mathbf{q}(0) = 0$, we get

$$A_j = - \frac{\mathbf{q}^{(j)\mathrm{T}} \cdot \mathbf{F}}{\omega_j^2 - \lambda^2}. \tag{2.47}$$

Moreover, if ω_j increases rapidly with j, then A_j tends to zero (the numerator remains bounded in view of (2.39)).

Thus, upon such an approach, the nodal moment is the solution of the generalized problem for eigenvalues:

$$K\mathbf{q} = \omega^2 M\mathbf{q}. \tag{2.37}$$

The complexity of this problem lies in that we have to deal with more than one matrix. However, this problem can be reduced to a problem involving one matrix:

$$M^{-1}K\mathbf{q} = \omega^2\mathbf{q}. \tag{2.37$'$}$$

However, in this case, we lose the main advantage of the matrices in the method of finite elements, i.e. their band-like nature. Moreover, the inversion of the matrix M is no longer a simple operation when its order is large. The methods for solving the generalized problem of eigenvalues by taking into consideration the properties of the method of finite elements (the banded matrices and the necessity of finding only a part of the spectrum) appeared comparatively recently. These are the methods of simultaneous iterations; their description and analysis can be found in [147].

Thus, the method of finite elements can be extended to the dynamic problems in the theory of elasticity, where the difficulties encountered are mainly computational in nature. This situation is typical of the method of finite elements; although the method of solution is clear, the order of the resulting systems of equations quite often (for example, in the case of three-dimensional problems) imposes its restrictions, and not all the problems can be solved.

Section 3
Diffraction of an Elastic Wave at a Circular Disc

The method of finite differences can be applied for the solution of one axially symmetric problem in the theory of elasticity [148]. We shall proceed from Eq. (4.4″), Ch. 2, Vol. 1, taking into consideration the fact that the component u_φ

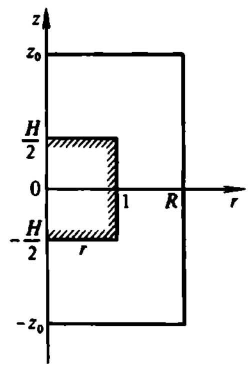

Fig. 86. Meridian cross section. Space with a cylindrical cavity, filled by a rigid inclusion.

is equal to zero, and the remaining components are independent of the coordinate φ:

$$\mu\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial r^2} + (1-\mu)\frac{\partial^2 w}{\partial r\,\partial z} + \frac{1}{r}\,\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial^2 w}{\partial z^2} + \mu\frac{\partial^2 u}{\partial r^2} + \frac{\mu}{r}\,\frac{\partial w}{\partial r} + (1-\mu)\left(\frac{\partial^2 u}{\partial z\,\partial r} + \frac{1}{r}\,\frac{\partial u}{\partial z}\right) = \frac{\partial^2 w}{\partial t^2}. \tag{3.1}$$

The components of the stress tensor are expressed in terms of the displacements and their derivatives as follows:

$$\sigma_z = \frac{\partial w}{\partial z} + (1-2\mu)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right), \qquad \tau_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right). \tag{3.2}$$

Next, the units of measurements are chosen in such a way that the velocity of propagation of the longitudinal waves and the density of the medium are equal to unity.

Suppose that in an elastic space we have a cylindrical cavity $-H/2 \leqslant z \leqslant H/2$, $r \leqslant 1$, which is filled by a rigid inclusion of density ρ (Fig. 86). From the positive side of the z-axis, we have an incident longitudinal wave having the form of a blurred step[5]. Under the action of this wave, the cylinder (i.e. the inclusion) is set into motion. We denote by w_0 the displacement of the cylinder along the z-axis. The equation of motion of the cylinder has the form

$$\rho\frac{H}{2}\,\ddot{w}_0 = \int_\Gamma (r\sigma_r dr + \tau_{rz} dz). \tag{3.3}$$

[5] The presence of a sharp step leads to the appearance of oscillations in the vicinity of the wavefront.

Here, we mean the values of the corresponding components of stresses at the boundary surface, and Γ is the contour of the cylinder in the meridian plane. We shall carry out all constructions in the meridian cross section. Let Q_1 and Q_2 be the displacements of the cylinder along the normal and the tangent to the contour. Obviously, $Q_1 = 0$ and $Q_2 = w_0$ at the lateral face, while on the end faces $Q_1 = w_0$ and $Q_2 = 0$. Further, let q_1 and q_2 be the displacements in the elastic medium at the boundary with the cylinder along the normal and the tangent to Γ.

Let us now describe the initial conditions. In front of the leading edge of the wave, the medium is at rest and free of load, while behind the trailing edge, the medium is in a state of uniaxial strain, and $\sigma_z = 1$. Then,

$$\begin{gathered} w_0(0) = \frac{dw_0(0)}{dt} = 0, \; u(0, r, z) = \frac{du(0, r, z)}{dt} = 0, \\ \frac{\partial w(0, r, z)}{\partial z} = \frac{\partial w(0, r, z)}{\partial t} = f\left[\frac{1}{T}\,(H + T - 2z)\right], \end{gathered} \tag{3.4}$$

where

$$f(\xi) = 0 \quad (\xi \geqslant 1), \qquad f(\xi) = 1 \quad (\xi \leqslant -1),$$
$$f(\xi) = \frac{1}{2}\,(1 + \xi^2 \operatorname{sgn} \xi) - \xi \qquad (-1 < \xi < 1).$$

On the contour Γ, we specify the following type of boundary conditions:

$$\begin{gathered} \frac{dq_2}{dt} = \frac{d\dot{Q}_2}{dt} \quad (|\tau_{rz}| < k), \\ \tau_{rz} = k \operatorname{sgn}\left(\frac{dq_2}{dt} - \frac{dQ_2}{dt}\right) \quad (|\tau_{rz}| = k), \end{gathered} \tag{3.5}$$

where k is a certain parameter. Thus, the attainment of a certain value by the shearing stresses results in slip.

We shall now go over to a description of the computational scheme. Here, a quite important question arises about the transition to a finite domain. It is suggested that a certain domain (whose cross section in the meridian plane is bounded by the contour Γ_1 (see Fig. 86), namely, $z = \pm z_0$, $r = R$) of sufficiently large dimensions be specified, so that the effect of the perturbations arising as a result of a transition to a finite domain may be eliminated (to a certain extent) by a suitable choice of boundary conditions. The starting point is the arguments given in [149] while considering the vibrations of a string of finite dimensions, where it has been shown that there are no reflected waves for certain boundary conditions. The solution obtained here will be identical to the solution obtained for the case of an infinite string.

In our case, firstly, we shall proceed from the fact that the boundary of Γ_1 is so far from the cylinder that the perturbed motion may be considered as one-dimensional. If we use the same notation as before (on the contour Γ), we arrive at the equations

$$\frac{\partial^2 q_1}{\partial t^2} = \frac{\partial^2 q_1}{\partial n^2}, \quad \frac{\partial^2 q_2}{\partial t^2} = \mu \frac{\partial^2 q_2}{\partial n^2}. \tag{3.6}$$

Here $\partial/\partial n$ denotes the derivative with respect to the normal to Γ_1. The boundary conditions at the segments $|z| = z_0$ of the boundary are given by

$$q_1 = w - f\left(t - z + \frac{H}{2}\right), \quad q_2 = u,$$

while on the segment $r = R$ of the boundary, the boundary conditions are given by

$$q_1 = u, \quad q_2 = w - f\left(t - z + \frac{H}{2}\right).$$

The general solution of Eqs. (3.6) contains a wave going off to infinity as well as a wave arriving from infinity. Since the latter type of waves must be absent, we introduce the constraint

$$\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial n} = 0, \quad \frac{\partial q_2}{\partial t} + \sqrt{\mu}\frac{\partial q_2}{\partial n} = 0. \tag{3.7}$$

In order to consider in a unique way the case when the cylinder has an infinitely low density, we modify Eq. (3.3)[6] by taking into consideration the energy contained in the layer adjacent to Γ and bounded by the contour Γ_2: $|z| = (1/2)(H + h)$, $r = 1 + h/2$:

$$\rho\frac{H}{2}\frac{d^2w_0}{dt^2} + \iint\limits_{D_1} \frac{\partial^2 w}{\partial t^2} r\,dr\,dz$$

$$= \int\limits_{\Gamma_2} \left(r\frac{\partial w}{\partial z}dr + \mu r\frac{\partial w}{\partial r}dz\right) + (1 + \mu)\left(1 + \frac{h}{2}\right)(u_a - u_b), \tag{3.8}$$

where D_1 is the domain contained between Γ and Γ_2; u_a and u_b are the values of u in the upper and lower corners of the contour Γ_2. Here, h is a constant which will later determine the mesh size. Let us consider the conditions arising on the symmetry axis:

$$u = 0, \quad \frac{\partial w}{\partial r} = 0. \tag{3.9}$$

The initial conditions (equal to zero) will be imposed for $t = H/2 - z_0 < 0$ (i.e. at the instant of the entrance of the wave into the domain of computation).

We divide the domain contained between the contours Γ and Γ_2 into squares with side h. For the sake of convenience, the step h is chosen in such a way that the nodes of the net are situated on the contours Γ and Γ_1 (in this case, $1/h$, H/h, z_0/h, and R/h will be integers).

---

[6] A modification is necessary in order to be able to carry out calculations for $\rho \ll 1$, since for $\rho = 0$, Eq. (3.3) degenerates.

The values of time were chosen with a certain step t_0, starting with $t = H/2 - z_0 + t_0$.

We shall be using the terms "the value in the middle layer (t)", "the value in the upper layer ($t + t_0$)" and "the value in the lower layer ($t - t_0$)". For approximating the derivatives at the internal points, central difference derivatives (14.3), Ch. 1, Vol. 1 were used. These can be written in a somewhat different form as follows:

$$\delta_z f_{ij} = \frac{1}{2h}(f_{i,j+1} - f_{i,j-1}),$$

$$\delta_{zz} f_{ij} = \frac{1}{h^2}(f_{i,j+1} - 2f_{i,j} + f_{i,j-1}), \tag{3.10}$$

$$\delta_{zr} f_{ij} = \frac{1}{4h^2}(f_{i+1,j+1} + f_{i-1,j-1} - f_{i+1,j-1} - f_{i-1,j+1}).$$

Then, we get the following difference analogues for Eqs. (3.1):

$$\begin{aligned} \delta_{tt}u &= \mu\delta_{zz}u + \delta_{rr}u + (1-\mu)\,\delta_{rz}w + \frac{1}{hi}\delta_r u - \frac{1}{(hi)^2}u, \\ \delta_{tt}w &= \delta_{zz}w + \mu\delta_{rr}w + \frac{\mu}{hi}\delta_r w + (1-\mu)\,\delta_{rz}u + \frac{1-\mu}{hi}\delta_z u. \end{aligned} \tag{3.11}$$

This is a three-layered pattern with second-order accuracy. For its stability, it is necessary that Courant's conditions be satisfied (see Sec. 14, Ch. 1, Vol. 1).

We shall now go over to the construction of the difference scheme for the boundary conditions. The difference analogue of the conditions (3.6) has the form

$$\delta_t q_1 + \delta_n q_1 = 0, \qquad \delta_t q_2 + \sqrt{\mu}\,\delta_n q_2 = 0. \tag{3.12}$$

Since the central differences are used, (3.12) will contain points lying outside the domain. In order to eliminate these points, we also use Eqs. (3.6). As a result, we get the following approximation for the boundary conditions:

$$\begin{aligned} &\frac{2}{h}(\delta_t q_1 + \delta_n q_1) + \delta_{tt} q_1 - \delta_{nn} q_1 = 0, \\ &\frac{2\sqrt{\mu}}{h}(\delta_t q_2 + \sqrt{\mu}\,\delta_n q_2) + \delta_{tt} q_2 - \mu\delta_{nn} q_2 = 0. \end{aligned} \tag{3.13}$$

For transforming the condition (3.3), we again use the central differences. This gives

$$\frac{\rho(H+h)}{2}\delta_{tt}w_0 = \sum_{\Gamma_2} r\,(\delta_z^{1/2}w + \mu\delta_r^{1/2}w)$$

$$+\frac{h}{2}\sum_{\Gamma_3}\delta_{tt}w+\frac{1-\mu}{2}\left(1+\frac{h}{2}\right)\left[u\left(t,1+h,h+\frac{H}{2}\right)\right.$$

$$\left.-u\left(t,1+h,-h-\frac{H}{2}\right)\right]+O(h^2). \qquad (3.14)$$

Here, Γ_3 is a segment of the straight line $r = 1$, $-H/2 < z < H/2$; the superscript 1/2 indicates that the approximation was carried out with a step $h/2$. Here, the computation of $\delta_{tt}w$ was made in the lower layer. For Eq. (3.3), we used an approximation of the first order[7]. For the remaining boundary conditions, an approximation having an accuracy of the second order was used.

The condition on the symmetry axis assumes the form

$$u = 0, \qquad \delta_r w = 0. \qquad (3.15)$$

The condition on the boundary of the cylinder $q_1 = Q_1$ remains unaltered, while the conditions (3.5) assume the form

$$\begin{aligned}\delta_t(q_2 - Q_2) &= 0 \quad \left(|\delta_n q_1| < \frac{k}{\mu}\right);\\ \delta_n q_2 &= \frac{k}{\mu}\,\mathrm{sgn}\,\delta_t(q_2 - Q_2) \quad \left(|\delta_n q_2| = \frac{k}{\mu}\right).\end{aligned} \qquad (3.16)$$

The solution of this non-linear system is of the form

$$\begin{aligned}q_2 &= -y_1 \text{ if } |y_1 + y_2| < k_2;\\ q_2 - y_2 &= k_2\,\mathrm{sgn}\,(y_1 + y_2) \text{ if } |y_1 + y_2| \geqslant k_2.\end{aligned} \qquad (3.17)$$

Here,

$$k_2 = \frac{2hk_1}{\mu},\quad y_1 = \frac{2}{3}\,h\delta_t^1(q_2 - Q_2) - q_2,\quad y_2 = \frac{2}{3}\,h\delta_t^1 q_2 + q_2.$$

Thus, the final stage now consists in just a direct determination of the displacements for the upper layer.

We shall now give the results of a number of calculations carried out for fixed parameters $h = 0.2$, $z_0 = 2.5$, and $R = 8.0$. Figure 87 shows the velocity $v(t)$, and the curves *1, 2, 3,* and *4* correspond to $\rho = 0.5$, 1, 2, and 4 respectively, where $H = 1$, $\mu = 0.3$, $T = 0.5$, and $k = 0$. Figure 88 shows the stresses $\sigma_z(r)$ on the upper end face for the time of establishment of a steady state. Here. the curves *1, 2,* and *3* correspond to $H = 0.5$, 1, and 2, and $\mu = 0.3$, $\rho = 1$, $k = 0$.

The dependence $\sigma_z(t)$ for the stresses at the centre of the upper end face is shown in Fig. 89. The curves *1* and *2* correspond to $\mu = 0.1$ and 0.5, for $H = 2$, $\rho = 1$, and $T = 0.25$.

[7] A higher-order approximation would lead to the implicit approximation (3.3).

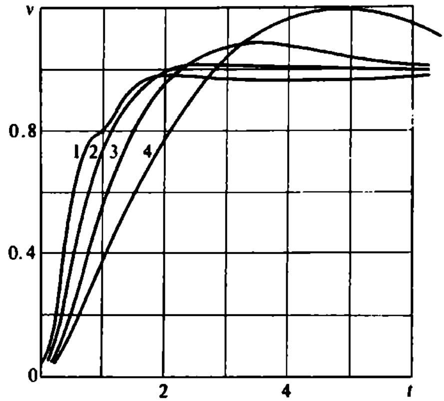

Fig. 87. The velocity $v(t)$ for different values of the parameter ρ.

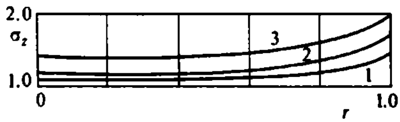

Fig. 88. The stress σ_z at the upper end face.

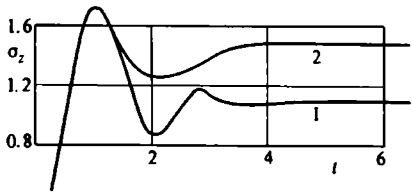

Fig. 89. Dependence of the stress σ_z on time at the centre of the upper end face.

Section 4
Propagation of Elastic Waves in a Circular Cylinder

Let us consider the axially symmetric dynamic problem in the theory of elasticity for a circular cylindrical body [150]. Let R_0 be the radius of the cylinder, l its length, and z the axis of rotation. We rewrite the equations of motion (4.2), Ch. 2, Vol. 1,

in a somewhat modified form:

$$
\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial v_r}{\partial t}, \\
&\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho \frac{\partial v_z}{\partial t},
\end{aligned}
\tag{4.1}
$$

where v_r and v_z are the components of the velocity vector.

Similarly, we rewrite Hooke's law (3.30″), Ch. 2, Vol. 1 in a modified form, having differentiated it with respect to time:

$$
\begin{aligned}
&\frac{\partial v_r}{\partial t} = \frac{1}{2\mu}\left[\frac{\partial \sigma_r}{\partial t} - \left(1 - \frac{2\mu}{3\lambda + 2\mu}\right)\frac{\partial p}{\partial t}\right], \\
&\frac{v_r}{r} = \frac{1}{2\mu}\left[\frac{\partial \sigma_\theta}{\partial t} - \left(1 - \frac{2\mu}{3\lambda + 2\mu}\right)\frac{\partial p}{\partial t}\right], \\
&\frac{\partial v_z}{\partial t} = \frac{1}{2\mu}\left[\frac{\partial \sigma_z}{\partial t} - \left(1 - \frac{2\mu}{3\lambda + 2\mu}\right)\frac{\partial p}{\partial t}\right], \\
&\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} = \frac{1}{\mu}\frac{\partial \tau_{rz}}{\partial t}, \qquad p = \frac{1}{3}(\sigma_r + \sigma_\theta + \sigma_z).
\end{aligned}
\tag{4.2}
$$

As a result, we get a closed system of equations in stresses and velocities.

We start from the following initial and boundary conditions. For $t = 0$, all stresses and velocities are identically equal to zero. The lateral surface is free of stresses. On the end face $z = 0$, the velocity $v_z \equiv v_0$ and $\tau_{r\theta} \equiv 0$. For $z = l$, the stress vector vanishes. The most interesting facts in this problem are associated with the propagation of waves, hence, in order to numerically solve this problem, it is expedient to use the method of characteristic surfaces, proposed in [151]. The use of the characteristic surfaces allows us to completely describe the wave propagation and at the same time causes the least spreading of the abrupt change in the solution at the fronts of the propagating waves.

We can reduce the systems (4.1) and (4.2) obtained above to dimensionless form by introducing the following variables:

$$
\begin{aligned}
&\tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_0}, \quad \tilde{v}_i = \frac{v_i}{v_0}, \quad \tilde{t} = \frac{ta}{R_0}, \quad \tilde{r} = \frac{r}{R_0}, \quad \tilde{z} = \frac{z}{R_0}, \\
&\sigma_0 = \rho a v_0, \quad a^2 = \frac{\lambda + 2\mu}{\rho}, \quad \beta = \frac{2(1 - \nu)}{1 - 2\nu}, \quad \alpha = \frac{\beta}{2(1 + \nu)}.
\end{aligned}
\tag{4.3}
$$

Henceforth, the tilde over the dimensionless quantities will be omitted. We then get a system which can be represented in the matrix form:

$$A_t \frac{\partial \mathbf{u}}{\partial t} + A_r \frac{\partial \mathbf{u}}{\partial r} + A_z \frac{\partial \mathbf{u}}{\partial z} = \mathbf{B}, \tag{4.4}$$

where $\mathbf{u}$ is the vector of the solution with components $\mathbf{u}(v_r, v_z, \sigma_r, \sigma_\theta, \sigma_z, \tau_{rz})$, and $\mathbf{B}$ is the vector of the right-hand sides, $\mathbf{B}(-(\sigma_r - \sigma_\theta)/r, -\tau_{rz}/r, 0, v_r/r, 0, 0)$, A_t, A_r, A_z are symmetric matrices of the following type:

$$A_t = \left[\begin{array}{cc|ccc|c} -1 & 0 & & & & \\ 0 & -1 & & 0 & & 0 \\ \hline & & \alpha & -\alpha\nu & -\alpha\nu & \\ & 0 & -\alpha\nu & \alpha & -\alpha\nu & 0 \\ & & -\alpha\nu & -\alpha\nu & \alpha & \\ \hline & 0 & & 0 & & -\beta \end{array}\right],$$

$$A_r = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & & & & \\ 0 & 0 & & 0 & & 0 \\ 0 & 0 & & & & \\ \hline 0 & 1 & & 0 & & 0 \end{array}\right], \qquad A_z = \left[\begin{array}{cc|ccc|c} 0 & & 0 & 0 & 0 & 1 \\ & & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & & & & \\ 0 & 0 & & 0 & & 0 \\ 0 & 1 & & & & \\ \hline 1 & 0 & & 0 & & 0 \end{array}\right].$$

We shall get an equation for the characteristic surfaces $\varphi(r, z, t) = 0$ for the system under consideration. This equation can be written as follows:

$$\left| A_t \frac{\partial \varphi}{\partial t} + A_r \frac{\partial \varphi}{\partial r} + A_z \frac{\partial \varphi}{\partial z} \right| = 0. \tag{4.5}$$

This equation can be written in a somewhat different, and more convenient, form:

$$|A_r n_r + A_z n_z - DA_t| = 0, \tag{4.5'}$$

where $n_i = \left(\dfrac{\partial \varphi}{\partial x_i}\right) \Big/ |\mathrm{grad}\, \varphi|$ are the direction cosines of the normal n to the characteristic surface $\varphi(r, z, t) = 0$, and $D = -\left(\dfrac{\partial \varphi}{\partial t}\right) \Big/ |\mathrm{grad}\, \varphi|$ is the velocity of propagation of the characteristic surface $\varphi = 0$ in the rz space.

After the determinant has been evaluated, Eq. (4.5′) assumes the form

$$[D^2 - (n_r^2 + n_z^2)]\left[D^2 - \frac{1}{\beta}(n_r^2 + n_z^2)\right] D^2 = 0. \qquad (4.5'')$$

It follows from this that the conical surface (4.5″) breaks up into two circular cones and a straight line which may be seen as a degenerate cone. It should be noted that unlike the case of one wave equation (see Sec. 9, Ch. 1, Vol. 1), there are three circular cones in this case. A longitudinal wave propagating with a velocity $D_p = \pm 1$ corresponds to the first cone, while the second cone has a transverse wave corresponding to it, travelling with a velocity $D_s = \pm\beta^{-1/2}$. A stationary wave with $D_{st} = 0$ corresponds to the degenerate cone.

We shall find the compatibility conditions on the characteristic surfaces. For this purpose, we determine the so-called left eigenvectors ω of the matrix $A = A_r n_r + A_z n_z$ with respect to the matrix A_t: [8]

$$\omega A - \omega D A_t = 0. \qquad (4.6)$$

Each eigenvalue D has a corresponding one-parametric set of left eigenvectors, which can be found in an explicit form from Eq. (4.6) and have the following components:

$$\begin{aligned}
&\omega_p\left(n_r, n_z, 1 - \frac{2}{\beta}n_z^2,\ 1 - \frac{2}{\beta},\ 1 - \frac{2}{\beta}n_r^2,\ \frac{2}{\beta}n_r n_z\right),\\
&\omega_s\left(-n_z, n_r, -2\frac{n_r n_z}{\beta^{1/2}}, 0, \frac{2}{\beta^{1/2}}n_r n_z, \frac{n_r^2 - n_z^2}{\beta^{1/2}}\right),\\
&\omega_{st}^{(1)}\ (0, 0, n_z^2, 0, n_r^2, 0),\\
&\omega_{st}^{(2)}\ (0, 0, 0, 1, 0, 0).
\end{aligned} \qquad (4.7)$$

Multiplying the original system of equations (4.4) by each of the eigenvectors (4.7), we get the equations of compatibility on the characteristic surfaces:

$$\omega_p A_t\left(\frac{\partial \mathbf{u}}{\partial t} + n_r\frac{\partial \mathbf{u}}{\partial r} + n_z\frac{\partial \mathbf{u}}{\partial z}\right) + \omega_p K\left(n_r\frac{\partial \mathbf{u}}{\partial z} - n_z\frac{\partial \mathbf{u}}{\partial r}\right) = \omega_p\cdot\mathbf{B}, \qquad (4.8)$$

$$\omega_s A_t\left[\frac{\partial \mathbf{u}}{\partial t} + \beta^{-1/2}\left(n_r\frac{\partial \mathbf{u}}{\partial r} + n_z\frac{\partial \mathbf{u}}{\partial z}\right)\right] + \omega_s K\left(n_r\frac{\partial \mathbf{u}}{\partial z} - n_z\frac{\partial \mathbf{u}}{\partial r}\right) = \omega_s\cdot\mathbf{B}, \qquad (4.9)$$

$$\omega_{st}^{(1)} A_t\frac{\partial \mathbf{u}}{\partial t} + \omega_{st}^{(1)} K\left(n_r\frac{\partial \mathbf{u}}{\partial z} - n_z\frac{\partial \mathbf{u}}{\partial r}\right) = \omega_{st}^{(1)}\cdot\mathbf{B}, \qquad (4.10)$$

$$\omega_{st}^{(2)} A_t\frac{\partial \mathbf{u}}{\partial t} + \omega_{st}^{(2)} K\left(n_r\frac{\partial \mathbf{u}}{\partial z} - n_z\frac{\partial \mathbf{u}}{\partial r}\right) = \omega_{st}^{(2)}\cdot\mathbf{B}, \qquad (4.11)$$

---

[8] If the matrix A_t is unitary, vector ω is just an eigenvector of matrix A.

where

$$K = n_r A_z - n_z A_r.$$

For example, let us consider the relation (4.8), which is valid on the cone of longitudinal waves. This relation contains derivatives with respect to two variables, s and l. The variable s changes along the bicharacteristic (i.e. the line of contact of the characteristic plane with the normal n and the longitudinal wave cone), while the variable l changes along the direction orthogonal to it:

$$\frac{\partial}{\partial s_i} = \frac{\partial}{\partial t} + D_i\left(n_r\frac{\partial}{\partial r} + n_z\frac{\partial}{\partial z}\right), \qquad \frac{\partial}{\partial l} = n_r\frac{\partial}{\partial z} - n_z\frac{\partial}{\partial r}.$$

These transformations enabled us to go over from differential equations containing three variables to relations with only two variables. Let us write the relations (4.8)-(4.11) in the curvilinear coordinates n and l, associated with the characteristic cones. In these coordinates, the equations assume a simpler form:

$$\frac{\partial}{\partial s_1}(v_n + \sigma_n) + \frac{\partial}{\partial l}\left(\tau_{nl} + \frac{\nu}{1-\nu}v_l\right) = \omega_p \cdot \mathbf{B}, \tag{4.8'}$$

$$\frac{\partial}{\partial s_2}(\beta^{1/2}\tau_{nl} + v_l) + \frac{\partial}{\partial l}(\sigma_l + \beta^{-1/2}v_n) = \omega_s \cdot \mathbf{B}, \tag{4.9'}$$

$$\frac{\partial}{\partial t}\left(\sigma_n - \frac{3\nu}{1+\nu}p\right) - \frac{2}{\beta}\frac{\partial v_l}{\partial l} = 0, \tag{4.10'}$$

$$\frac{\partial}{\partial t}\left(\sigma_\theta - \frac{3\nu}{1+\nu}p\right) - \frac{2}{\beta}\frac{v_r}{r} = 0. \tag{4.11'}$$

Here, v_n and v_l are the components of the velocity in the n and l directions respectively, and

$$\sigma_n = \sigma_r n_r^2 + 2\tau_{rz}n_r n_z + \sigma_z n_z^2,$$
$$\sigma_l = \sigma_r n_z^2 + \sigma_z n_r^2 - 2n_r n_z \tau_{rz},$$
$$\tau_{nl} = (\sigma_z - \sigma_r)\, n_r n_z + \tau_{rz}(n_r^2 - n_z^2)$$

are the components of the stress tensor.

Beyond this, the problem consists in the choice, from a multitude of these relations, of six linearly independent relations on the number of unknown functions. The choice of such a system of characteristic equations may not be uniquely made if the number of variables is more than two. It is natural to choose them in such a way that the equations obtained as a result of difference discretization are the simplest, allow us to use the regular net, and satisfy the required Courant's stability condition.

Relations (4.8')-(4.11') assume the simplest form when the normal n coincides with the normal to the coordinate planes. This corresponds to the values $n_r = 0$, $n_z = \pm 1$ and $n_z = 0$, $n_r = \pm 1$.

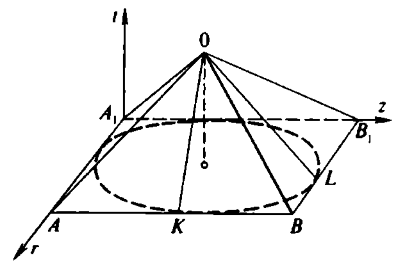

Fig. 90. An elementary cell.

In order to obtain a regular net and to avoid intermediate interpolations, the following choice of characteristic relations is proposed for finding the solution at the internal points. We apply four relations (4.8′) to the longitudinal wave cone for $n_r = 0$, $n_z = \pm 1$ and $n_z = 0$, $n_r = \pm 1$. These planes form a right tetrahedral pyramid described around the cone and are tangential to it along the four bicharacteristics. The base of the pyramid is a square whose sides coincide with the coordinate lines (Fig. 90). The other two equations are taken in the characteristic planes (for the degenerate cone) passing through the pyramid axis and the diagonals on its base ($n_r = 1/\sqrt{2}, n_z = \pm 1/\sqrt{2}$). These six relations form a linearly independent system of equations, equivalent to the original system. This system of equations may be represented in the following form:

$$\begin{aligned}
&\frac{\partial}{\partial s_i}(v_r \pm \sigma_r) \pm \frac{\partial}{\partial z}\left(\tau_{rz} \pm \frac{\nu}{1-\nu} v_z\right) = \mp \frac{\sigma_r - \sigma_\theta}{r} - \frac{\nu}{1-\nu}\frac{v_r}{r},\\
&\frac{\partial}{\partial \eta_i}(v_z \pm \sigma_z) \pm \frac{\partial}{\partial r}\left(\tau_{rz} \pm \frac{\nu}{1-\nu} v_r\right) = \mp \frac{\tau_{rz}}{r} - \frac{\nu}{1-\nu}\frac{v_r}{r},\\
&\frac{\partial}{\partial t}\left[6\nu\alpha p - \frac{\beta}{2}(\sigma_r + \sigma_z - 2\tau_{rz})\right] + \frac{\partial}{\partial l_i}(v_r \mp v_z) = 0,
\end{aligned} \tag{4.12}$$

where

$$\frac{\partial}{\partial s_i} = \frac{\partial}{\partial r} \mp \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial \eta_i} = \frac{\partial}{\partial z} \mp \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial l_i} = \frac{\partial}{\partial z} \mp \frac{\partial}{\partial r}.$$

As mentioned above, each of these relations contains derivatives with respect to only two variables. This circumstance has a favourable effect on numerical integration. Equations (4.12) may be represented in the form of integral relations, and the integration is carried out over the area of the face of the pyramid, on which it is satisfied. With the help of Green's formula, we transform the surface integral into the contour integral. This allows us to obtain integral relations which do not contain any derivatives. For example, the first of the relations (4.12) assumes the form

$$\int_C (v_r \pm \sigma_r)\,dz \pm \left(\tau_{rz} \pm \frac{\nu}{1-\nu} v_z\right) ds_i = \mp \int_\Sigma \left(\frac{\sigma_r - \sigma_\theta}{r} \pm \frac{\nu}{1-\nu}\frac{v_r}{r}\right) d\Sigma, \tag{4.13}$$

where Σ is the area of the triangle AOB, and C is its contour. With the help of some quadratic formula, we get a finite-difference representation for Eq. (4.13). The remaining relations in (4.12) are transformed in a similar way.

We break up the domain, occupied by a body in the rz plane at the instant $t = kh$, by a square net with step h. The nodal points of this net are denoted by ih, jh. At the next instant of time $t + \Delta t = (k + 1)h$, the solution will be obtained at the nodal points of a net which is displaced by half a step in r and z. The coordinates of this net will be given by $(i + 1/2)h$, $(j + 1/2)h$. For such a net, Eq. (4.13) may be written in the following form using, for example, the trapezium formula:

$$a^{k+1}_{i+1/2,\, j+1/2} = \frac{1}{2}\left[a^k_{i+1,j} + a^k_{i+1,j+1} + c^k_{i+1,j+1} - c^k_{i+1,j} + \frac{h}{3}\left(L^k_{1_{i+1/2,j+1}} + L^k_{1_{i+1,j}} + L^{k+1}_{1_{i+1/2,j+1/2}}\right)\right], \tag{4.14}$$

$$b^{k+1}_{i+1/2,j+1/2} = \frac{1}{2}\left[b^k_{i,j+1} + b^k_{i,j} + d^k_{i,j} - d^k_{i,j+1} + \frac{h}{3}\left(L^k_{2_{i,j+1}} + L^k_{2_{i,j}} + L^{k+1}_{2_{i+1/2,j+1/2}}\right)\right],$$

where

$$v_r = \frac{1}{2}(a + b), \quad \sigma_r = \frac{1}{2}(a - b), \quad L_{1,2} = \pm\frac{\sigma_r - \sigma_\theta}{r} - \frac{\nu}{1-\nu}\frac{v_r}{r}.$$

From these two equations, we can determine v_r and σ_r.

It should be observed that the system (4.14) of the finite-difference equations in the unknowns a^{k+1}, b^{k+1}, ... has a diagonal form. This results in a considerable simplification of computations (as compared to a direct application of the difference method to the original differential equations).

We shall now take up the construction of a difference diagram for the boundary points. In the general case, the boundary conditions can be written in the form

$$a_1 v_\nu + b_1 \sigma_\nu = f_1, \qquad a_2 v_\tau + b_2 \tau_{\tau\nu} = f_2, \tag{4.15}$$

where the subscripts ν and τ correspond to the normal and the tangent to the boundary.

It should be noted that the use of characteristic equations at the boundary for constructing difference diagrams is quite significant. As a matter of fact, while constructing the difference analogue of Eqs. (4.12) for the boundary points, we obtain an overdeterminate problem, since the number of relations is larger than the number of unknowns. A canonical characteristic form of the equations with respect to the boundary permits us to obtain uniquely the correct difference diagram for calculations of the boundary points. In order to construct this diagram, we should write down all the characteristic equations (4.8′)-(4.11′) with the normal n coinciding with the normal ν to the boundary. In this case, we get four relations, one for each of the formulas (4.8′)-(4.11′), and to these we add the two boundary conditions. Thus we get six relations in all.

By way of an example, let us consider the lateral surface ($r = R_0$). Equations (4.8′)-(4.11′) for $n_r = 1$ and $n_z = 0$ correspond to the canonical form. As in the case of an internal point, we replace them by integral equations, integrating over the areas of the triangles AOB, A_1OB_1, and A_2OB_2, lying respectively in the tangential planes to the cones of longitudinal waves, transverse waves, and to the boundary. As a result, we obtain the following finite-difference equations.

For $\triangle AOB$, the first of the relations (4.14) will be valid. Integrating the relation (4.9′) over the area of the triangle A_1OB_1, we get

$$(\beta^{1/2}\tau_{rz} + v_z)_O = \frac{1}{2}\,[(\beta^{-1/2}\tau_{rz} + v_z)_{B_1}] + \frac{\beta^{-1/2}}{2}\,[(\beta^{1/2}\sigma_z + v_r)_{B_1}]$$

$$-\frac{h\beta^{-1/2}}{3(1 + \beta^{-1/2})}\left[\left(\frac{\tau_{rz}}{r}\right)_{A_1} + \left(\frac{\tau_{rz}}{r}\right)_{B_1} + \left(\frac{\tau_{rz}}{r}\right)_O\right]. \qquad (4.16)$$

For the triangle A_2OB_2, we get

$$q_O = \frac{1}{6}\,(q_{A_2} + q_{B_2} + 4q_O) - \frac{1 - 2\nu}{2(1 - \nu)^2}\,[(v_z)_{A_2} - (v_z)_{B_2}]$$

$$+ \frac{h(1 - 2\nu)\nu}{4(1 - \nu)^2}\left[\left(\frac{v_r}{r}\right)_{A_2} + \left(\frac{v_r}{r}\right)_{B_2} + \left(\frac{v_r}{r}\right)_O\right],$$

where

$$q = \sigma_z - \frac{\nu}{1 - \nu}\,\sigma_r.$$

The corner points are calculated from the same relations as used for the boundary points, independently for each of the surfaces forming the angle. The values obtained in this way are then averaged.

Let us now analyze the stability of the difference diagram constructed above. Obviously, Courant's condition is automatically satisfied, but in this case, it is only a necessary condition. A complete investigation of stability has been carried out for a two-dimensional axially symmetric problem in [150].

We shall clarify this by taking the example of a one-dimensional problem.

We construct the particular solutions of the one-dimensional problem in the following form:

$$v = \bar{v}e^{ik_1lh}\,e^{\rho n\Delta t} \qquad (\lambda = e^{\rho\Delta t}).$$

We then get a homogeneous algebraic system of equations for $\bar{v}$:

$$(\lambda - e^{ik_1h})\,\bar{a} = 0, \qquad (\lambda - e^{-ik_1h})\,\bar{b} = 0,$$

$$2(\lambda - \cos k_1h)\,\bar{\sigma}_z + \frac{\nu i}{1 - \nu}\sin k_1h\,(\bar{a} + \bar{b}) = 0,$$

$$\left(\lambda - \frac{1}{\lambda}\right)\bar{c} + \frac{2i}{\alpha}(\bar{a} + \bar{b})\sin k_1 h = 0, \tag{4.17}$$

$$\bar{c} = 6\bar{p}\nu - \frac{\beta}{\alpha}(\bar{\sigma}_r + \bar{\sigma}_z),$$

$$(\lambda - \cos k_1 h)\bar{v}_z + \frac{2\alpha}{\beta} i \sin kh\bar{\tau}_{rz} = 0,$$

$$\left(\lambda - \frac{1}{\lambda}\right)\bar{\tau}_{rz} - \frac{2i}{\alpha}\bar{v}_z \sin kh = 0.$$

The condition that the determinant be equal to zero leads to an equation for λ. The solutions of Eqs. (4.17) will be bounded only if the following condition is satisfied:

$$\max_i \lambda_i \leqslant 1.$$

This is the Neumann condition. We shall prove that this condition really holds. As a matter of fact, the system (4.17) breaks up into two, the first in terms of the quantities $\bar{a}$, $\bar{b}$, $\bar{\sigma}_z$, and $\bar{c}$, and the second in terms of $\bar{v}_z$ and $\bar{\tau}_{rz}$.

The characteristic numbers of the first system are

$$\lambda_1 = e^{ik_1h}, \qquad \lambda_2 = e^{-ik_1h}, \qquad \lambda_3 = \cos k_1 h, \qquad \lambda_{4,5} = \pm 1.$$

If follows from this that harmonic waves corresponding to the roots λ_1 and λ_2 propagate without damping and dispersion, and hence the values of σ_r and v_r on the fronts of the longitudinal cylindrical discontinuous waves will not be blurred, while σ_z will be smoothened.

For the second system, which characterizes the change in the values of τ_{rz} and v_z in transverse cylindrical waves, the characteristic equation has the form

$$\lambda^3 - \lambda^2 \cos k_1 h - \lambda\left(1 - \frac{2}{\beta}\sin^2 k_1 h\right) + \cos k_1 h = 0.$$

The roots of this equation satisfy the condition $\max_i |\lambda_i| < 1$. Hence a blurring of the discontinuity will take place on the front of a transverse wave.

It should be remarked that if there are two velocities of propagation no difference diagram can describe both the waves without blurring in the problem considered here about the impingement of a wave on the end face of a cylinder with a constant velocity $v_r = V_0$. The most significant are the longitudinal waves, and hence it is expedient to describe such waves with the greatest accuracy.

The computational method described above was applied for solving the above-formulated problem about the impingement of a wave on the end face with a constant velocity.

Figure 91 shows the variation of the longitudinal velocity v_z along the cylinder for $r = 0$ at different instants of time t. The length of the cylinder is sufficiently large, and hence there is no reflection of the waves from the end $z = l$. In the example considered above, the values $V_0 = 5$, and $\nu = 0.3$ were taken. Until the arrival of

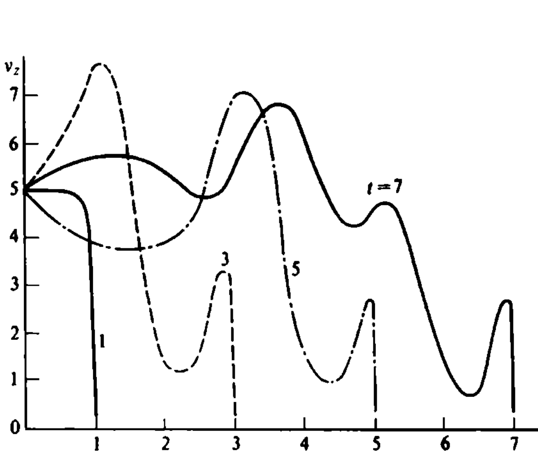

Fig. 91. Distribution of velocity v_z for an infinite cylinder.

the waves reflected from the lateral surface of the cylinder, the longitudinal wave propagates without any distortion and has a clearly defined step form, whose front moves with a velocity $a = D_p = ((\lambda + 2\mu)/\rho)^{1/2}$. With the arrival of the lateral waves, the pattern of the pulse propagation is complicated, even for $\bar{t} = 3R/c$, the pulse disintegrates into its forerunner, a short-duration pulse, propagating in the form of a soliton[9] with velocity a, and the fundamental pulse, whose front propagates with a velocity close to $a_0 = (E/\rho)^{1/2}$, obtained from the theory of bars. The amplitude of the forerunner is somewhat less than the applied velocity and changes very little with time. The fundamental pulse has the form of a step on which oscillations are superimposed. The maximum value of the velocity is attained behind the front and exceeds the applied value of V_0 by some 40-45%. For this problem, an asymptotic solution is known [152, 153] for $t \gg R_0/c$.

The numerical solution obtained for this problem is qualitatively identical to this solution for $t \sim 10R_0/c$. The maximum increase of v_z over the applied V_0 is about 30%.

Figure 92 shows the distribution of velocities along the axis $r = 0$ in a rod of finite length $l = 5R_0$ after reflection of the longitudinal wave from the free end face of the cylinder for different instants of time. The velocity after reflection at the free end rapidly increases and approaches the value predicted in the elementary theory of bars. Qualitatively, the same picture is observed for other values of r also, but the amplitude of oscillations due to lateral waves decreases with distance from the axis. The stress σ_z at the contact surface at the point $r = z = 0$ decreases from the value $\rho a V_0$ to the value $\rho a_0 V_0$, obtained from the theory of bars, and then oscillates

[9] This is how an isolated wave is called in the wave theory.

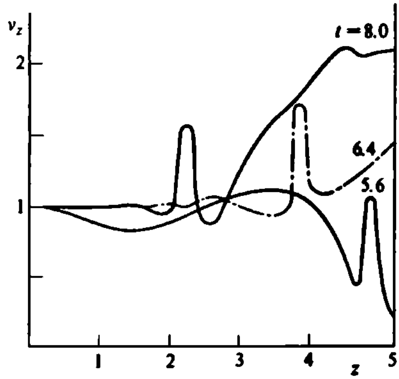

Fig. 92. Distribution of velocity v_z for a finite cylinder.

about this value with a period of oscillations near the value $4R_0/a$ in this case.

Section 5
Diffraction of an Elastic Wave at a Spherical Cavity

Suppose that we have a spherical cavity of radius r_0 in space. A plane longitudinal wave of intensity σ_0 arrives from infinity. We shall consider the interaction of this wave with the spherical cavity [154]. Naturally, we shall carry out the analysis in the spherical system of coordinates (r, θ, φ), choosing the axis $\theta = 0$ so that the problem is axially symmetric.

We introduce dimensionless quantities

$$\sigma'_{ij} = \frac{\sigma_{ij}}{\lambda + 2\mu}, \qquad u'_i = \frac{u_i}{a}, \qquad r' = \frac{r}{r_0}, \qquad t' = \frac{ta}{r_0}. \tag{5.1}$$

Let us write down the system of dymanic equations in the theory of elasticity for stresses and velocities (for the sake of convenience, we shall omit the primes), choosing the units of measurements such that $a = 1$ and $\rho = 1$:

$$\dot{\sigma}_r = u_{,r} + (1 - 2\mu')\frac{v_{,r}}{r} + (1 - 2\mu')\left(\frac{2u}{r} + \frac{v \cot\theta}{r}\right),$$

$$\dot{\sigma}_\theta = (1 - 2\mu')u_{,r} + \frac{v_{,\theta}}{r} + \frac{2(1 - \mu')u}{r} + \frac{(1 - 2\mu')\, v \cot\theta}{r},$$

$$\dot{\sigma}_\varphi = (1 - 2\mu')\left(u_{,r} + \frac{v_{,\theta}}{r}\right) + \frac{2(1 - \mu')\,u}{r} + \frac{v \cot\theta}{r}, \tag{5.2}$$

$$\dot{\tau} = \mu'\left(\frac{u_{,\theta}}{r} + v_{,r} - \frac{v}{r}\right),$$

$$\dot{u} = \sigma_{r,r} + \frac{\tau_{,\theta}}{r} + \frac{2\sigma_r - \sigma_\theta - \sigma_\varphi + \tau\cot\theta}{r},$$

$$\dot{v} = \tau_{,r} + \frac{\sigma_{\theta,\theta}}{r} + \frac{3\tau + (\sigma_\theta - \sigma_\varphi)\cot\theta}{r}$$

$$\left(\mu' = \frac{b^2}{a^2} = \frac{\mu}{\lambda + 2\mu}\right).$$

We shall now write down the initial and the boundary conditions. In the perturbed region (before the interaction of the wave with the cavity), the stresses and the velocities are given by

$$\begin{aligned} \sigma_r &= -\sigma_0(1 - 2\mu'\sin^2\theta)f(\xi), & \tau &= \sigma_0\mu' f(\xi)\sin 2\theta,\\ \sigma_\theta &= -\sigma_0(1 - 2\mu'\cos^2\theta)f(\xi), & u &= -\sigma_0 f(\xi)\cos\theta,\\ \sigma_\varphi &= -\sigma_0(1 - 2\mu')f(\xi), & v &= \sigma_0 f(\xi)\sin\theta, \end{aligned} \tag{5.3}$$

where $f(\xi)$ is a function characterizing the form of the incident wave, and is a given function (probably discontinuous). In the remaining part, there are no velocities and stresses. Assuming that there are no stresses in the cavity, we get

$$\sigma_r = 0, \qquad \tau = 0 \qquad (r = 1). \tag{5.4}$$

We go over to new functions which are introduced with the help of the formulas

$$p = \frac{\sigma_r + \sigma_\theta}{2}, \quad q = \frac{\sigma_r - \sigma_\theta}{2}, \quad \Phi = \sigma_\varphi - \frac{1 - 2\mu'}{1 - \mu'}p. \tag{5.5}$$

The system of equations can then be written (by introducing the matrix notation) as a symmetric system of the first order

$$A^t_{y,t} + A^r_{y,r} + A^\theta_{y,\theta} + \mathbf{B} = 0. \tag{5.6}$$

Here, A^t is a positive definite matrix, A^r and A^θ are symmetric matrices which are written as follows:

$$y = \begin{bmatrix} u \\ v \\ p \\ q \\ \Phi \\ \tau \end{bmatrix}, \qquad A^t = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{1-\mu'} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\mu'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{\mu'} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{\mu'} \end{bmatrix},$$

$$A^r = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.6'}$$

$$A^\theta = \frac{1}{r}\begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \frac{1}{r}\begin{bmatrix} 0 & 0 & m_1 & -3 & -s & 1 \\ 0 & 0 & m_1 s & s & 0 & s \\ m_2 & m_3 s & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ m_4 & m_4 s & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$m_1 = \frac{\mu'}{1-\mu'}, \quad m_2 = \frac{3\mu' - 2}{1-\mu'}, \quad m_3 = \frac{2\mu' - 1}{1-\mu'}, \quad m_4 = \frac{4\mu' - 3}{1-\mu'},$$

$$s = \cot\theta.$$

We shall seek the solution in the form of the sum of two terms, one of which is associated with the incident wave, and the other with the reflected wave. All the quantities pertaining to the reflected wave carry a subscript "1". Then, we get the following boundary conditions for the vector $\mathbf{y}_1$:

$$\begin{aligned} p_1 + q_1 &= (1 - 2\mu' \sin^2\theta) f(\xi), \\ \tau_1 &= -\mu' \sin 2\theta\, f(\xi). \end{aligned} \tag{5.7}$$

Obviously, the initial conditions are equal to zero.

Keeping in view the construction of the difference solution, we must restrict the computational domain by an additional (outer) boundary, in the same way as was done in Sec. 3. For this purpose, it is convenient to choose a sphere of sufficiently large radius $r = R_1$. The boundary conditions on this sphere can then be written in the form

$$p_1 + q_1 + u_1 = 0, \qquad \tau_1 + \sqrt{\mu}\,'v_1 = 0. \tag{5.8}$$

In view of the axial symmetry, the following conditions must be satisfied on the axis:

$$\tau = 0, \quad v = 0, \quad p_{,\theta} = q_{,\theta} = f_{,\theta} = u_{,\theta} = 0. \tag{5.9}$$

We shall now describe the computational scheme. We carry out the discretization of the domain into cells

$$r_i = (1 + \Delta\theta)^i \quad (i = 0, 1, \dots, M), \qquad \theta_j = \Delta\theta j \quad (j = 0, 1, \dots, N).$$

The varying step Δr_i (along the coordinate r) is chosen in such a way that the sides of the cell are nearly equal.

The functions p, q, τ, and f will be determined only at nodal points of the net, while u and v will be determined at the centres of the cells. In order to increase the accuracy of the approximation in time[10], we shall determine the velocities u and v for time $t = \Delta t(k - 0.5)$, while p, q, τ, and f will be determined for time $t = \Delta tk$ $(k = 0, 1, \dots, K)$.

While constructing difference diagrams for internal points, central differences in coordinates and time (the so-called space cross) are used.

Since the boundary of the computational domain passes through the nodal points of the net, there are no conditions imposed on the velocity. The conditions (5.9) are replaced by

$$\tau = 0, \quad \delta_\theta p = \delta_\theta q = \delta_\theta f = 0. \tag{5.10}$$

Here, δ_θ is the operator approximating the first derivative with respect to the angle θ, with a second-order accuracy[11].

The conditions (5.7) are not sufficient for determining the four quantities p, q, τ, and f in the upper layer in time for $r = 1$. Hence we add to these two bicharacteristic relations for the system (5.2). As a result, we get the conditions

$$\begin{gathered} p + q = (1 - 2\mu' \sin^2\theta) f(\xi), \qquad \tau = -\mu' f(\xi) \sin 2\theta, \\ \mu' p_{,t} - (1 - \mu') q_{,t} - 2\mu'(1 - \mu')\frac{1}{r} v_{,\theta} \\ -\mu'(3 - 4\mu')\frac{u}{r} - \mu'(1 - 2\mu')\frac{v}{r} \cot\theta = 0, \end{gathered} \tag{5.11}$$

---

[10] In this case, it is possible to use the central difference in time, which leads to second-order accuracy.

[11] $\delta_\theta f = [4f(\theta_0 + \Delta\theta) - f(\theta_0 + 2\Delta\theta) - 3f(\theta_0)]/2\Delta\theta$.

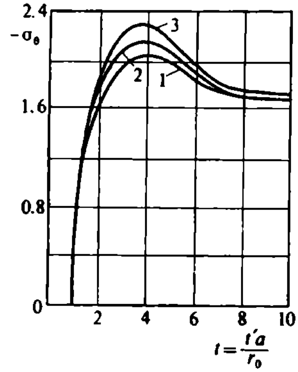

Fig. 93. Time dependence of σ_θ.

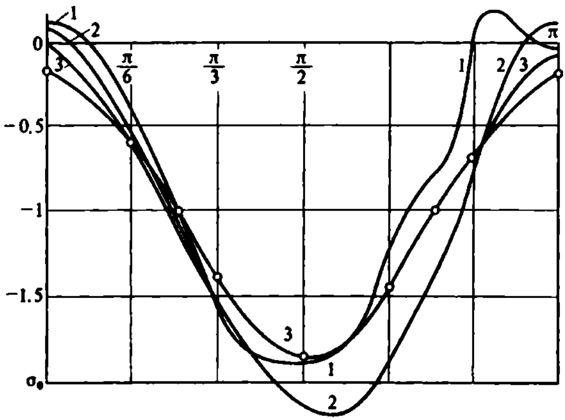

Fig. 94. Angular distribution of σ_θ.

$$f_{,t} - \mu' \frac{3 - 4\mu'}{1 - \mu'} \left(\frac{u}{r} + \frac{v}{r} \cot \theta \right) = 0.$$

To the conditions at the boundary $r = R_1$, also,we have added the bicharacteristic relations.

In order to eliminate the oscillation of the numerical solution while computing in the vicinity of the wavefront, we used three-point "smoothing" with coefficients depending on the gradient of the "smoothened" solution (see, for example, [155]).

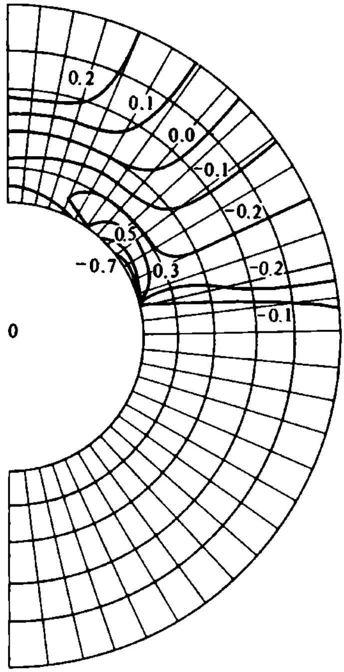

Fig. 95. The level lines for $|\tau_{max}|$.

We shall give the results of a number of calculations. Figure 93 shows the dependence of the stress σ_θ on time at the equatorial point. The curves *1*, *2*, and *3* correspond to the values $\mu' = 0.25$, 0.3, and 0.33 respectively.

Figure 94 shows the distribution of stresses $\sigma_\theta(\theta)$ at different instants of time. The curves *1*, *2*, and *3* correspond to the instants $t = 2.094$, 4.188, and 6.983. The curve with the points is the solution of the static problem.

Figure 95 shows the level lines for $|\tau_{max}|$ corresponding to $t = 0.837$.

Appendices

1. Kelvin-Somilyana Solution for an Anisotropic Body

The potential theory can be used for solving three-dimensional problems in the theory of elasticity for the case of a general type of anisotropy. In order to construct the corresponding integral equations, it is necessary to use (as in the case of an isotropic body) the Kelvin-Somilyana solution.

We shall now describe a method of constructing such a solution. We start from Hooke's law (3.15), Ch. 2, Vol. 1. After substituting these relations into Eqs. (4.4), Ch. 2, Vol. 1, we get equilibrium equations in displacements for an anisotropic medium:

$$c_{pqrs}u_{r,sq} + P_p = 0. \tag{1}$$

In order to construct the solutions of these equations over the entire space, we apply Fourier transformations (4.17), Ch. 1, Vol. 1 for all three variables. Then, for the transforms, we get

$$k_{pr}(\lambda)\bar{u}_r(\lambda) = \bar{P}_p(\lambda), \tag{2}$$

where

$$k_{pr}(\lambda) = c_{pqrs}\lambda_q\lambda_s.$$

Let $k^*_{pr}(\lambda)$ be the elements of a matrix, which is inverse[1] to the matrix with elements k_{pr}, i.e.

$$k^*_{pt}(\lambda)k_{pr}(\lambda) = \delta_{tr}. \tag{3}$$

Multiplying Eqs. (2) by the matrix with elements $k^*_{pq}(\lambda)$, we at once arrive at the following expression for the displacement transform:

$$\bar{u}_q(\lambda) = k^*_{pq}(\lambda)\bar{P}_p(\lambda). \tag{4}$$

Applying Fourier's inverse transformation, we get the expressions for the displacements:

$$(2\pi)^3u_q(x) = \int P_p(y)\,dy_1\,dy_2\,dy_3 \int e^{i[\lambda_\alpha(y_\alpha - x_\alpha)]}k^*_{pq}(\lambda)\,d\lambda_1\,d\lambda_2\,d\lambda_3. \tag{5}$$

---

[1]Construction of the inverse matrix is possible on account of the fact that the energy of deformation is a positive definite form.

The representation obtained above can be simplified if we consider that it is necessary to construct a solution for the concentrated force of unit magnitude (in this case, only one component is different from zero). We have

$$(2\pi)^3 u_{pq}(x) = \int e^{-i\lambda_\alpha x_\alpha} k^*_{pq}(\lambda)\, d\lambda_1\, d\lambda_2\, d\lambda_3. \tag{6}$$

Let us now go over from the integral over the entire space to the integral over a sphere of unit radius by introducing the coordinates η ($\lambda_p = \rho\eta_p$). Moreover, it should be noted that the elements $k_{pq}(\lambda)$ of the matrix are homogeneous functions of the second order, and correspondingly, the elements $k^*_{pq}(\lambda)$ of the matrix will have an order -2. Hence,

$$(2\pi)^3 u_{pq}(x) = \lim_{R\to\infty} \int k^*_{pq}(\eta)\, d\omega \int_{\rho=0}^{R} \cos(\rho\eta_\alpha x_\alpha)\, d\rho$$

$$= \lim_{R\to\infty} \int k^*_{pq}(\eta) \frac{\sin R\eta_\alpha x_\alpha}{\eta_b x_b}\, d\omega. \tag{7}$$

We now introduce the coordinates θ, φ on the sphere, choosing the direction $\theta = 0$ such that it coincides with the direction of the vector x_p. In this case, the expression (7) can be further simplified:

$$(2\pi)^3 r u_{pq}(x) = \lim_{R\to\infty} \int_0^{2\pi} d\varphi \int_{-1}^{1} k_{pq}(\mu, \theta) \frac{\sin Rr\mu}{\mu}\, d\mu. \tag{8}$$

Here,

$$r^2 = x_\alpha x_\alpha, \qquad \mu = \cos\theta, \qquad k_{pq}(\mu, \theta) = k^*_{pq}(\eta).$$

Considering Eq. (4.14), Ch. 1, Vol. 1, we get

$$(2\pi)^3 r u_{pq}(x) = \pi \int_0^{2\pi} k_{pq}(0, \varphi)\, d\varphi. \tag{9}$$

This representation can be written in a more convenient form:

$$u_{pq}(x) = \frac{1}{8\pi^2 r} \oint k^*_{pq}(\eta)\, ds. \tag{10}$$

Here, the integral is taken over the arc of the circle whose plane is perpendicular to the vector x_p, whose centre coincides with the origin of coordinates, and whose radius is equal to unity.

If the medium is transversally anisotropic[2], the integrals in (10) are evaluated in an explicit form.

---

[2]The medium having one plane of isotropy is called transversally anisotropic. In this case, the non-zero elastic constants are expressed in terms of five constants.

Let us also consider the plane problem in the theory of elasticity for an anisotropic body. Suppose that at every point of a plate, we have a plane of symmetry of elastic properties, parallel to the middle plane. As in the isotropic case (see Sec. 4, Ch. 3, Vol. 1), we shall assume that the forces applied to the ends of the plate act in the middle plane. Then, going over to quantities averaged over the thickness of the plate, we get the following relations between stresses and strains:

$$\begin{aligned} \varepsilon_x &= c_{11}\sigma_x + c_{12}\sigma_y + c_{16}\tau_{xy}, \\ \varepsilon_y &= c_{12}\sigma_x + c_{22}\sigma_y + c_{26}\tau_{xy}, \\ \gamma_{xy} &= c_{16}\sigma_x + c_{26}\sigma_y + c_{66}\tau_{xy}. \end{aligned} \tag{11}$$

As in the case of Eq. (4.20), Ch. 3, Vol. 1, we introduce Airy's function $U(x, y)$. Then, after substituting stresses into the compatibility equations for strains in accordance with (11), we obtain the following equation for the Airy function:

$$c_{22}\frac{\partial^4 U}{\partial x^4} - 2c_{26}\frac{\partial^4 U}{\partial x^3\partial y} + (2c_{12} + c_{66})\frac{\partial^4 U}{\partial x^2\partial y^2} - 2c_{16}\frac{\partial^4 U}{\partial x\partial y^3} + c_{11}\frac{\partial^4 U}{\partial y^4} = 0. \tag{12}$$

As expected, Eq. (12) turns out to be biharmonic in the isotropic case.

Let μ_k be the roots of the characteristic equation

$$c_{11}\mu^4 - 2c_{16}\mu^3 + (2c_{12} + c_{66})\mu^2 - 2c_{26}\mu + c_{22} = 0. \tag{13}$$

It is well known (see, for example, [156]) that the roots of this equation are always complex (for an isotropic medium, they are found to be pure imaginary). Suppose that

$$\mu_1 = \alpha + i\beta, \qquad \mu_2 = \alpha - i\beta, \qquad \mu_3 = \gamma + i\delta, \qquad \mu_4 = \gamma - i\delta. \tag{14}$$

Then, the general solution of Eq. (12) for different roots may be expressed in the following form:

$$U(x, y) = u_1(z_1) + \overline{u_1(z_1)} + u_2(z_2) + \overline{u_2(z_2)}, \tag{15}$$

where $u_1(z_1)$ and $u_2(z_2)$ are functions of complex variables $z_1 = x + \mu_1 y$ and $z_2 = x + \mu_2 y$.

The expressions for the stresses and displacements are

$$\begin{aligned} \sigma_x &= 2\mathrm{Re}\,[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)], \\ \sigma_y &= 2\mathrm{Re}\,[\Phi_1'(z_1) + \Phi_2'(z_2)], \\ \tau_{xy} &= -2\mathrm{Re}\,[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)], \\ u &= 2\mathrm{Re}\,[p_1\Phi_1(z_1) + p_2\Phi_2(z_2)], \\ v &= 2\mathrm{Re}\,[q_1\Phi_1(z_1) + q_2\Phi_2(z_2)], \end{aligned} \tag{16}$$

where

$$\Phi_1(z_1) = \frac{du_1}{dz_1}, \qquad \Phi_2(z_2) = \frac{du_2}{dz_2},$$

$$p_1 = c_{11}\mu_1^2 + c_{12} - c_{16}\mu_1^2, \qquad p_2 = c_{11}\mu_3^2 + c_{12} - c_{16}\mu_3,$$

$$q_1 = c_{12}\mu_1 + \frac{c_{22}}{\mu_1} - c_{26}, \qquad q_2 = c_{12}\mu_3 + \frac{c_{22}}{\mu_3} - c_{26}.$$

2. On an Approach for Viscoelastic Media

A viscoelastic medium is one whose rheology is described by the equations

$$\sigma_{ij}(t) = E_{ijkl}(t)\varepsilon_{kl}(t) \qquad (i,j = 1,2,3), \tag{1}$$

where

$$E_{ijkl}\varepsilon_{kl} = C_{ijkl}\varepsilon_{kl}(t) + \int_0^t B_{ijkl}(t,\tau)\varepsilon_{kl}(\tau)d\tau. \tag{2}$$

The integral operators in (2) are often called hereditary-type operators (generally speaking, the integrals in (2) are taken in the sense of Stieltjes integrals). We consider the equality

$$E_{ijkl} = E_{klij} = E_{klji} = E_{lkij}, \tag{3}$$

which is also valid for the components C_{ijkl} and B_{ijkl}. If $B_{ijkl} \equiv 0$, the relations (1) are equivalent to Hooke's law.

We shall confine ourselves to one particular dependence for the functions $B_{ijkl}(t,\tau)$ (this dependence is encountered for several materials which are important from a practical point of view):

$$B_{ijkl}(t,\tau) = B_{ijkl}(t-\tau). \tag{4}$$

We shall be considering the so-called quasistatic problems in the theory of viscoelasticity, in which the boundary conditions (in displacements or stresses) may vary with time, and the inertial terms are negligibly small. We shall assume that on the part S_1 of the boundary surface, the displacements $\mathbf{F}_1(y,t)$ are given, while on the remaining part S_2, we are given the stresses $\mathbf{F}_2(y,t)$. Naturally, it is possible to specify only displacements or only stresses throughout. However, it is necessary to assume for the mixed problem that the contour, which is the boundary of the surfaces S_1 and S_2, remains unchanged during deformation.

Thus, the solution of the problem of viscoelasticity consists in the solution of the equilibrium equations (4.4), Ch. 3, Vol. 1, the strain compatibility equations, and the equation of state (1) under the constraint (4). If we carry out the Laplace transformation (4.30), Ch. 1, Vol. 1 in time in the equilibrium equations, the strain compatibility equations, and the equation of state, we arrive at relations between stresses and strains in terms of transforms. For example, the equilibrium equations can be reduced to the following system of equations:

$$\frac{\partial\bar{\sigma}_{ij}(p,x)}{\partial x_j} = 0. \tag{5}$$

Exactly in the same way, the equations following from the strain compatibility conditions will formally coincide with the original equations where we replace $\varepsilon_{ij}(x, t)$ by $\bar{\varepsilon}_{ij}(p, x)$. The boundary conditions in transforms are given by

$$\begin{aligned} \bar{\mathbf{u}}(p, x) &= \bar{\mathbf{F}}_1(p, x) \qquad (x \in S_1), \\ T_\nu \bar{\mathbf{u}}(p, x) &= \bar{\mathbf{F}}_2(p, x) \qquad (x \in S_2). \end{aligned} \tag{6}$$

According to the convolution theorem (see Sec. 4, Ch. 1, Vol. 1), the relations between σ_{ij} and ε_{ij} will assume the form

$$\bar{\sigma}_{ij}(p, x) = D_{ijkl}(p)\bar{\varepsilon}_{kl}(p, x), \tag{7}$$

where

$$D_{ijkl}(p) = C_{ijkl} + \bar{B}_{ijkl}(p). \tag{8}$$

In this way, we get equations for the transforms of stresses and strains, which are completely identical to the equations in the theory of elasticity. True, these equations contain a parameter whose different values will correspond to different values of elastic constants (called the instantaneous moduli) in the auxiliary problem of the theory of elasticity. After the problem has been solved in terms of transforms (more precisely, we solve a class of problems for such values of the parameter p, which are intended to be used in the inverse transformation), it is necessary to restore the required quantities. Naturally, the problem is simplified if its solution in terms of the transforms can be obtained in an explicit form.

The method described above (called Volterra's principle) may be formulated in the following way. In order to solve a problem in the theory of viscoelasticity, it is necessary to solve an ordinary problem in the theory of elasticity, treating the operators as constant numbers. As a result, the solution will be expressed in the form of a product of a function depending on the elastic constants and the coordinates and a known function of time. At the final stage, we must go over from the elastic constants to the operators.

3. On the Phisically Non-linear Theory of Elasticity

A characteristic feature in the behaviour of a number of materials is that even for small values of strain a departure is observed from the linear dependence between stresses and strains[3]. We shall confine ourselves to the case when the dependence between stresses and strains can be represented in the form

$$\varepsilon_{ij} = \frac{k(\sigma)}{3K}\sigma\delta_{ij} + \frac{g(T_i^2)}{2G}(\sigma_{ij} - \sigma\delta_{ij}), \tag{1}$$

where K is the bulk modulus of strain, $k(\sigma)$ and $g(T_i)$ are functions of hydrostatic pressure and the shearing stress intensity. Besides, in accordance with the ex-

[3]This type of non-linearity is called physical non-linearity to distinguish it from geometrical non-linearity, for which finite strains are taken into consideration.

perimental results which establish a practically linear dependence between the hydrostatic pressure and compressibility, we put $k(\sigma) = 1$, and for the sake of simplicity of analysis, we confine ourselves to the case

$$g(T_i^2) = 1 + gT_i^2,$$

where g = const.

Thus, the behaviour of the elastic medium under consideration will be characterized by three constants. The values of these three quantities for the case of copper are given below by way of an example

$$K = 13.44 \cdot 10^{10}\,\mathrm{N/m^2}, \qquad G = 4.51 \cdot 10\,\mathrm{N/m^2}, \qquad g = 0.18 \cdot 10^6 G^{-2}.$$

Let us consider the solution of plane problems. The following relations can be obtained from (1) for the case of plane stress state:

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{3K - 2G}{6KG}\sigma\delta_{ij} - \frac{g}{2G}T_i^2(\sigma_{ij} - \sigma\delta_{ij}), \tag{2}$$

where

$$T_i^2 = \frac{2}{9}[(\sigma_x + \sigma_y)^2 + 3(\tau_{xy}^2 - \sigma_x\sigma_y)].$$

For the case of plane strain, we get the following equality from the condition $\varepsilon_z = 0$ by neglecting the powers of T_i higher than 2:

$$\sigma_z = \frac{1}{2}\left[\frac{3K - 2G}{3K - G} + \frac{9KG}{(3K + G)^2}gT_i^2\right](\sigma_x + \sigma_y). \tag{3}$$

With the help of (3), we can transform (1). This leads to the expression

$$\varepsilon_{ij} = \frac{1}{2G}[\sigma_{ij} - a\sigma\delta_{ij} + gT_i^2(\sigma_{ij} - b\sigma\delta_{ij})], \tag{4}$$

which differs from (2) only in the values of the constants, and

$$T_i^2 = \frac{2}{9}[(1 + c)(\sigma_x + \sigma_y)^2 + 3(\tau_{xy}^2 - \sigma_x\sigma_y)],$$

$$a = \frac{3(3K - 2G)}{2(3K + G)}, \quad b = \frac{27K^2 + 18KG - 6G^2}{2(3K + G)^2}, \quad c = \frac{8G^2 - 6KG - 9K^2}{4(3K + G)^2}.$$

Thus, the solution of the problem for a physically non-linear elastic medium is reduced to the solution of the equilibrium equations (4.4), Ch. 3, Vol. 1, and of the strain compatibility equations (4.6), Ch. 3, Vol. 1, if we take into consideration the relations (4). Obviously, the plane strain problem and the problem of the plane stress state can be considered (as in the case of a linear medium) in the same way, since the difference lies only in the values of the constants.

Following [156], we introduce Airy's function $U(x, y)$ with the help of the same relations (4.20), Ch. 3, Vol. 1. The equilibrium equations will then be identically satisfied, while the strain compatibility equation can be transformed to a fourth-order linear equation in Airy's function by substituting in it the value of stresses in accordance with (4).

We carry out a transition to functions of a complex variable. For this purpose, we represent the relations (4.20), Ch. 3, Vol. 1 in the form

$$\sigma_x + \sigma_y = 4\frac{\partial^2 U}{\partial z\,\partial \bar{z}}, \qquad \sigma_x - \sigma_y + 2i\tau_{xy} = -4\frac{\partial^2 U}{\partial \bar{z}^2}. \tag{5}$$

Using relations (4), we obtain the following representation for the strain components:

$$\begin{aligned} \varepsilon_x + \varepsilon_y &= \frac{2}{G}\left[(1-2a)\frac{\partial^2 U}{\partial z\,\partial \bar{z}} + (1-2b)gT_i^2\frac{\partial^2 U}{\partial z\,\partial \bar{z}}\right], \\ \varepsilon_x - \varepsilon_y + 2i\gamma_{xy} &= \frac{2}{G}\left(-\frac{\partial^2 U}{\partial \bar{z}^2} - gT_i^2\frac{\partial^2 U}{\partial \bar{z}^2}\right). \end{aligned} \tag{6}$$

Taking into consideration the identity

$$2\frac{\partial}{\partial \bar{z}}(u + iv) = \varepsilon_x - \varepsilon_y + 2i\gamma_{xy},$$

we can integrate the second of the equations (6):

$$2G(u + iv) = -2\frac{\partial U}{\partial \bar{z}} + f(z) - 2g\int T_i^2\frac{\partial^2 U}{\partial \bar{z}^2}\,d\bar{z}, \tag{7}$$

where $f(z)$ is an arbitrary function of a complex variable. Next, we differentiate (7) with respect to z. This gives

$$2G\frac{\partial(u + iv)}{\partial z} = -2\frac{\partial^2 U}{\partial z\partial \bar{z}} + f'(z) - 2g\frac{\partial}{\partial z}\int T_i^2\frac{\partial^2 U}{\partial \bar{z}^2}\,d\bar{z}. \tag{8}$$

Using the identity

$$2\frac{\partial(u + iv)}{\partial z} = \varepsilon_x + \varepsilon_y + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

and the first of the equalities in (6), we can transform (8):

$$4\frac{\partial^2 U}{\partial z\,\partial \bar{z}} = \sigma_x + \sigma_y = 2[\varphi'(z) + \overline{\varphi'(z)} + A(z, \bar{z})], \tag{9}$$

where

$$A(z, \bar{z}) = -\frac{g}{1-a}\,\mathrm{Re}\left[(1-2b)T_i^2\frac{\partial^2 U}{\partial z\,\partial \bar{z}} + \frac{\partial}{\partial z}\int T_i^2\frac{\partial^2 U}{\partial \bar{z}^2}\,d\bar{z}\right],$$

$$\varphi'(z) = \frac{f'(z)}{8(1-a)}.$$

Equation (9) contains the real part only. Integrating this equation, we get

$$2\frac{\partial U}{\partial \bar{z}} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \int A(z,\bar{z})\,dz, \tag{10}$$

where $\psi(z)$ is an arbitrary function of a complex variable. With the help of (7) and (10), we get the following representation for displacements and stresses:

$$\begin{gathered} 2G(u+iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + 2B(z,\bar{z}), \\ \sigma_x - \sigma_y + 2i\tau_{xy} = -2\left[z\overline{\varphi''(z)} + \overline{\psi'(z)} + \int \frac{\partial A}{\partial z}\,dz\right], \end{gathered} \tag{11}$$

where

$$\varkappa = 3 - 4a, \qquad B(z,\bar{z}) = -g\int T_i^2\,\frac{\partial^2 U}{\partial \bar{z}^2}\,d\bar{z} - \frac{1}{2}\int A\,dz.$$

These expressions can be used to formulate the boundary value problems. For the case of the first problem, we at once get the boundary condition from (11):

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = f(t). \tag{12}$$

Thus, in order to solve the first problem[4], it is necessary to determine two analytical functions $\varphi(z)$ and $\psi(z)$ in the region occupied by the elastic body, such that these functions must satisfy the non-linear boundary condition (12). The boundary condition for the second problem is a little more complicated to form[5], starting from the representations (9) and (11):

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + \int A\,dz = f(t), \tag{13}$$

where, as in the case of the linear problem,

$$f(t) = \int_{s_0}^{s} (\sigma_{x\nu} + i\sigma_{y\nu})\,ds.$$

The solution of the boundary value problems is carried out by successive approximations. For this purpose, the conditions (12) and (13) should be expressed in the form

$$\varkappa\varphi^{(n)}(t) + \overline{t\varphi^{(n)\prime}(t)} - \overline{\psi^{(n)}(t)} = f(t) - 2B^{(n-1)}(t,\bar{t}), \tag{14}$$

$$\varphi^{(n)}(t) + \overline{t\varphi^{(n)\prime}(t)} + \overline{\psi^{(n)}(t)} = f(t) - \int A^{(n-1)}\,dz. \tag{15}$$

Thus, the solution of the non-linear boundary value problem has been reduced to a set of solutions of the linear problems. To a first approximation, there are no integrals on the right-hand sides, and the linear problem can be solved. Next, we determine the corrections to the right-hand side and the linear problem is solved once again, and so on.

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[4]For the sake of simplicity, we take into consideration only simply connected domains.

[5]The line of arguments is the same as followed for the linear problem (see Ch. 3, Vol. 1).

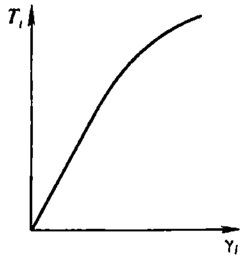

Fig. 96. Strain hardening curve.

In [156], for example, some problems on the concentration of stresses around a hole have been solved by using the method described above.

4. Method of Elastic Solutions in the Theory of Small Elasto-plastic Deformations

We shall describe the method of "elastic solutions" [157], applied for solving the problems in the theory of plasticity within the framework of the theory of small elasto-plastic deformations.

The main points of this theory are the following. As before, let σ_{ij} and ε_{ij} be the components of the stress tensor and the strain tensor respectively. These components are connected by the following relations:

$$\varepsilon_{ii} - \varepsilon = \frac{\psi(\gamma)}{2G}(\sigma_{ii} - \sigma), \qquad \varepsilon_{ij} = \frac{\psi(\gamma)}{G}\sigma_{ij} \qquad (i \neq j). \tag{1}$$

Here, $\varepsilon = \varepsilon_{ii}$, $\sigma = (1/3)\sigma_{ii}$, and $\psi(\gamma)$ is a certain function, determined experimentally (Fig. 96),

$$\gamma = \sqrt{1/3} \times \sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + (1/3)(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)}.$$

The coefficient in these equations has been chosen in such a way that the case $\psi = 1$ corresponds to an elastic medium. It should be observed that the same dependence holds between ε and σ as for the case of an elastic medium:

$$\varepsilon = \frac{3(1 - 2\nu)}{E}\sigma. \tag{2}$$

We transform (1) in such a way that the stresses are directly expressed in terms of the strains only:

$$\sigma_{ii} = \left(\frac{E}{1 - 2\nu} - \frac{2G}{\psi}\right)\varepsilon + \frac{2G}{\psi}\varepsilon_{ii},$$

$$\sigma_{ij} = \frac{G}{\psi}\varepsilon_{ij} \qquad (i \neq j). \tag{3}$$

In accordance with the boundary value problem in the theory of plasticity[6], we formulate the boundary value problem in the theory of elasticity for the domain occupied by the original body. Moreover, we require that the displacements (and hence the strains as well) be identical. We shall show that such an approach is possible. We denote the stresses in the elastic medium by σ_{ij}^* and write down the expressions for Hooke's law as follows:

$$\begin{aligned} \sigma_{ii}^* &= \left(\frac{E}{1-2\nu} - 2G\right)\varepsilon + 2G\varepsilon_{ii}, \\ \sigma_{ij}^* &= G\varepsilon_{ij} \qquad (i \neq j). \end{aligned} \tag{4}$$

From (3) and (4), we get the following expressions for the difference between the components $\sigma_{ij}^0 = \sigma_{ii}^* - \sigma_{ii}$:

$$\begin{aligned} \sigma_{ij}^0 &= 2G\left(1 - \frac{1}{\psi}\right)(\varepsilon_{ii} - \varepsilon), \\ \sigma_{ij}^0 &= G\left(1 - \frac{1}{\psi}\right)\varepsilon_{ij} \qquad (i \neq j). \end{aligned} \tag{5}$$

Going over on the right-hand side to the stresses in an elastic medium, we get

$$\sigma_{ii}^0 = \left(1 - \frac{1}{\psi}\right)(\sigma_{ii}^* - \sigma^*), \qquad \sigma_{ij}^0 = \left(1 - \frac{1}{\psi}\right)\sigma_{ij}^*. \tag{6}$$

Notice that the superscript "*" on σ may be omitted in view of (2).

We shall now write down the expressions for the stress components σ_{ij} in terms of the stresses for the auxiliary problem:

$$\sigma_{ii} = \frac{1}{\psi}\sigma_{ii}^* + \left(1 - \frac{1}{\psi}\right)\sigma, \qquad \sigma_{ij} = \frac{1}{\psi}\sigma_{ij}^*. \tag{7}$$

It can be shown that similar dependences hold for the stress intensities τ:

$$\tau = \tau^* - \tau^0, \qquad \tau^0 = G\left(1 - \frac{1}{\psi}\right)\gamma, \qquad \tau = \frac{1}{\psi}\tau^*. \tag{8}$$

Substituting into the equilibrium equations for the components their representations (7), we get

$$\sigma_{ij,i}^* + P_j = 0, \qquad P_j = -\sigma_{ij,i}^0. \tag{9}$$

Obviously, the terms P_j may be treated as fictitious body forces.

The expanded version of the expression for the force P_1 can be written as

---

[6]For the theory of plasticity considered here, the formulation of the boundary value problems is the same as in the theory of elasticity.

$$P_1 = -\left\{\frac{\partial}{\partial x}\left[2G\left(1 - \frac{1}{\psi}\right)(\varepsilon_x - \varepsilon_y)\right]\right.$$
$$\left. + \frac{\partial}{\partial y}\left[G\left(1 - \frac{1}{\psi}\right)\gamma_{yx}\right] + \frac{\partial}{\partial z}\left[G\left(1 - \frac{1}{\psi}\right)\gamma_{zx}\right]\right\}. \tag{10}$$

The following representation is also valid:

$$P_1 = -\left\{\frac{\partial}{\partial x}\left[\left(1 - \frac{1}{\psi}\right)(\sigma_x^* - \sigma)\right]\right.$$
$$\left. + \frac{\partial}{\partial y}\left[\left(1 - \frac{1}{\psi}\right)\tau_{yx}^*\right] + \frac{\partial}{\partial z}\left[\left(1 - \frac{1}{\psi}\right)\tau_{zx}^*\right]\right\}. \tag{11}$$

Thus, it has been proved that the auxiliary elastic problem must contain additional body forces[7].

Let us now consider the boundary conditions. On the part of the surface where the displacements are given, the problem is automatically solved—according to the construction of the auxiliary problem, the boundary conditions must be retained without any alterations. On the same part of the surface where stresses are given, we must introduce additional surface forces $\sigma_{\nu i}^0$, which are defined by Eqs. (7).

We shall give an expression for just one component (the remaining components are written in accordance with the formulas of identical structure):

$$\sigma_{\nu x}^0 = 2G\left(1 - \frac{1}{\psi}\right)\left[(\varepsilon_x - \varepsilon)l + \frac{1}{2}\gamma_{xy}m + \frac{1}{2}\gamma_{yz}n\right], \tag{12}$$

or,

$$\sigma_{\nu x}^0 = \left(1 - \frac{1}{\psi}\right)[(\sigma_x^* - \sigma)l + \tau_{xy}^* m + \tau_{xz}^* n]. \tag{12'}$$

Thus, the formulation of the problem (transition to the auxiliary elastic medium) has been accomplished in principle, although its actual solution is reduced to the solution of a non-linear system of differential equations.

We shall solve this system by the method of successive approximations. For the first approximation, we solve the problem of the theory of linear elasticity (in the absence of additional body forces and surface forces). The values of the function ψ are determined from the values of strain found in this problem. This allows us to find the additional body forces (in the second approximation) in accordance with (10), and the surface forces (in the second approximation) in accordance with (12). Next, we again solve the problem of the theory of elasticity, but in this case we consider the body forces and the modified surface forces: we find the strains, the function $\psi(\gamma)$, and then carry on the algorithm. It has been shown [158] that, under specific conditions ($\psi'(\gamma) > 0$) and for a sufficiently smooth function $\psi(\gamma)$, the process of successive approximations converges.

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[7]If body forces are present in the initial problem, they must be added to the fictitious body forces in the auxiliary problem.

We shall make one remark concerning the numerical realization of the method of elastic solutions. Since we must construct the solution corresponding to the body force which is specified with the help of its values at discrete points, it seems appropriate to use the apparatus of generalized elastic potentials (see Sec. 1, Ch. 3, Vol. 1). For such an approach, certain stresses appear on the surface, and these must be eliminated (in order to actually obtain a particular solution of the inhomogeneous equation with zero boundary conditions). This leads to the introduction of one more stage—the determination of these stresses and their inclusion (with the opposite sign) in the boundary condition for the next iteration.

After the process of elastic solutions has been completed, we can easily determine the state of stress at any point of the body. For this purpose, we must calculate the strains (i.e. the strains in the auxiliary elastic medium) and then obtain the stresses with the help of relations (3). Since, as a rule, computer programmes contain blocks which define stresses and not strains, it is worthwhile to calculate stresses in the elastic medium, and use the formulas (7) to go over to the real stresses.

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